Generic Laver diamonds at the continuum

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2025 年 11 月 12 日 (15:30 ∼17:00 JST) : **Kobe Set Theory Seminar**

▶ **2025 年 12 月 16 日 (10:45 ∼11:15 JST) : RIMS set theory workshop 2025**

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[0] This is an ongoing joint work with Francesco Parente.

Diamond at the continuum under Martin's Axiom

Proposition 1. MA + "$\aleph_1 < 2^{\aleph_0}$ is a successor cardinal" implies $\diamondsuit_{2^{\aleph_0}}$.

Proof. Trivial modulo a theorem by Shelah. □ (Proposition 1)

Theorem 2. (Shelah, [1]) If $\lambda = \chi^+ = 2^\chi > \aleph_1$ then $\diamondsuit_\lambda$ holds. □

Theorem 3. (Martin-Solovay) MA implies that $2^\kappa = 2^{\aleph_0}$ for all $\aleph_0 \leq \kappa < 2^{\aleph_0}$.

A sketch of Proof.

- For $\kappa < 2^{\aleph_0}$, let $\{a_\alpha : \alpha < \kappa\}$ be an almost disjoint family of elements of $[\omega]^{\aleph_0}$.
- For $A \subseteq \kappa$, show that, under MA, that there is $a \subseteq \omega$, s.t. $A = \{\alpha \in \kappa : |a \cap a_\alpha| = \aleph_0\}$. □ (Theorem 3)

Corollary 4 (to Theorem 3). MA implies that the continuum is regular. □

Corollary 5 (to Proposition 1). PFA implies $\diamondsuit_{2^{\aleph_0}}$. □

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[1] Saharon Shelah, Diamonds, Proc. of the AMS, Vol.138, No.6, (2010), 2151–2161. See also A. Rinot's note.

Diamond at the continuum under Martin's Axiom (2/2)

- Does MA imply $\diamondsuit_{2^{\aleph_0}}$ when $2^{\aleph_0}$ is a limit cardinal?

- Note that, by Corollary 4 , MA + "$2^{\aleph_0}$ is a limit cardinal" implies that $2^{\aleph_0}$ is weakly inaccessible.

Proposition 6. (modulo some large cardinal)
MA + "$2^{\aleph_0}$ is a limit cardinal" + $\neg\diamondsuit_{2^{\aleph_0}}$ is consistent.

A sketch of Proof.

- It is known that under consistency of some large cardinal, $\neg\diamondsuit_\kappa$ for an inaccessible cardinal κ is consistent. Actually it is also known that $\neg\diamondsuit_\kappa$ for a Mahlo κ is consistent (H. Woodin).

- Starting with such κ, force MA + $\kappa = 2^{\aleph_0}$ by the standard ccc p.o.. Then $\neg\diamondsuit_\kappa$ is preserved. □ (Proposition 6)

Diamonds and Laver diamonds

- Suppose that κ is an uncountable regular cardinal.
- (Reminder) $\diamondsuit_\kappa$ is the assertion that there is a sequence ($\diamondsuit_\kappa$-sequence) $\langle a_\alpha : \alpha \in \kappa \rangle$, s.t. $a_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, and for any $X \subseteq \kappa$, $\{\alpha \in \kappa : X \cap \alpha = a_\alpha\}$ is stationary in κ.

- Laver diamond (also called Laver function) at κ for a notion LC of large cardinal is a mapping $f : \kappa \to V_\kappa$ s.t. for any set a, and $\lambda > \kappa$ there is an elementary embedding $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$ for some inner model $M \subseteq \mathsf{V}$ with the closure property corresponding to LC s.t. $\underline{j(f)(\kappa) = a}$.

- Laver diamond exists for most of large large cardinals (for supercompact, extendible, hyperhuge, etc).

Diamonds and Laver diamonds (2/2)

- The notion of Laver diamond works only for a very large large cardinal κ.

- If $f : \kappa \to V_\kappa$ is a Laver diamond at κ (for any notion of large cardinal) then $\langle a_\alpha : \alpha < \kappa \rangle$ defined by

  $$a_\alpha := \begin{cases} f(\alpha); & \text{if } f(\alpha) \subseteq \alpha; \\ \emptyset; & \text{otherwise.} \end{cases}$$

  is a $\diamondsuit_\kappa$-sequence (see a similar argument in the proof of Lemma 8).

- Cf.:

Theorem 7. (Kunen, see A. Kanamori [2]) If κ is subtle then $\diamondsuit_\kappa$ holds.

[2] Akihiro Kanamori, Diamonds, large cardinals, and ultrafilters, Contemporary Mathematics, Vol.69 (1988).

Generic Laver Diamond

- Suppose that $\mathcal{P}$ is a class of (two-step) iterable p.o.s, and LC a notion of large cardinal.
- For an uncountable regular cardinal κ, the generic Laver Diamond Principle $\diamond^{\mathcal{P},\mathrm{LC}}_{Laver,\kappa}$ is the assertion of the existence of $\diamond^{\mathcal{P},\mathrm{LC}}_{Laver,\kappa}$-sequence $f : \kappa \to V_\kappa$:

(*) for any set a, and $\lambda > \kappa$, there is $\mathbb{P} \in \mathcal{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ s.t. there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, j satisfing the closure property corresponding to LC, and $j(f)(\kappa) = a$.

- Laver diamond at κ is simply a $\diamond^{\{\{\mathbb{1}\}\},\mathrm{LC}}_{Laver,\kappa}$-sequence (for any notion LC of large cardinal).
- A similar generic version of Laver diamond has been studied by Matteo Viale and Sean Cox but mainly in connection with MM and its fragments.

Generic Laver Diamond (2/2)

Lemma 8. $\diamondsuit^{\mathcal{P},\mathrm{LC}}_{Laver,\kappa}$ implies $\diamondsuit_\kappa$.

Proof. Suppose that $f : \kappa \to V_\kappa$ is a $\diamondsuit^{\mathcal{P},\mathrm{LC}}_{Laver,\kappa}$-sequence.

- Suppose $X \subseteq \kappa$ $(X \in \mathrm{V})$. Then there are $\mathbb{P} \in \mathcal{P}$, $\mathbb{G}$, j, M as in the definition of $\diamondsuit^{\mathcal{P},\mathrm{LC}}_{Laver,\kappa}$-sequence, s.t. $M \models j(f)(\kappa) = X$.
- ▷ Then $j(\{\alpha < \kappa : f(\alpha) = X \cap \alpha\}) \ni \kappa$ since $j(X) \cap \kappa = X$. It follows that $\mathrm{V} \models \{\alpha < \kappa : f(\alpha) = X \cap \alpha\}$ is stationary in κ.
- Thus, letting

$$a_\alpha := \begin{cases} f(\alpha), & \text{if } f(\alpha) \subseteq \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

  $\langle a_\alpha : \alpha < \kappa \rangle$ is a $\diamondsuit_\kappa$-sequence. □ (Lemma 8)

- The same proof actually shows that $\diamondsuit^{\mathcal{P},\mathrm{LC}}_{Laver,\kappa}$ implies $\diamondsuit_\kappa(S)$ for $S := \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \mu\}$ for all $\omega \leq \mu < \kappa$.

Consistency of generic Laver Diamond

Proposition 9. Suppose that $\mathcal{P}$ is a Σ_2 transfinitely iterable class of p.o.s containing a p.o. which provably adds a new real. If κ is an extendible cardinal, then there is a $\mathbb{P}_\kappa \in \mathcal{P}$ s.t.

$$\Vdash_{\mathbb{P}_\kappa} \text{“} 2^{\aleph_0} = \kappa, \ \mathcal{P}\text{-LgLCA for extendible, and } \diamondsuit^{\mathcal{P},\mathit{extendible}}_{\mathit{Laver},2^{\aleph_0}} \text{ holds”}.$$

A sketch of Proof. ▶ Let κ be extendible, and $f : \kappa \to V_\kappa$ a Laver diamond for extendible at κ.

▷ Let $\vec{P} := \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an iteration in $\mathcal{P} \cap V_\kappa$ with the support suitable for $\mathcal{P}$ s.t. for $\beta < \kappa$:

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} \underset{\sim}{\mathbb{R}}_\beta, & \text{if } f(\beta) = \langle \underset{\sim}{\mathbb{R}}_\beta, \underset{\sim}{a}_\beta \rangle \text{ where } \underset{\sim}{\mathbb{R}}_\beta, \underset{\sim}{a}_\beta \text{ are } \mathbb{P}_\beta\text{-names and} \\ & \Vdash_{\mathbb{P}_\beta} \text{“} \underset{\sim}{\mathbb{R}}_\beta \in \mathcal{P} \text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing,} & \text{otherwise.} \end{cases}$$

Consistency of generic Laver Diamond (2/3)

Claim 9.1 $\mathbb{P}_\kappa$ is as desired.

$\vdash$ (a): $\Vdash_{\mathbb{P}_\kappa}$ "$\mathcal{P}$-LgLCA for extendible": See e.g. here [3] .

(b): $\Vdash_{\mathbb{P}_\kappa}$ "$\Diamond^{\mathcal{P},extendible}_{Laver,\kappa}$":

- Let $\underset{\sim}{g}$ be a $\mathbb{P}_\kappa$-name s.t. $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{g} : \kappa \to V_\kappa$" and

  If $f(\alpha) = \langle \underset{\sim}{\mathbb{R}}_\alpha, \underset{\sim}{a}_\alpha \rangle$ where $\underset{\sim}{\mathbb{R}}_\alpha$ and $\underset{\sim}{a}_\alpha$ are $\mathbb{P}_\alpha$-names, then $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{g}(\alpha) = \underset{\sim}{a}^*_\alpha$" where $\underset{\sim}{a}^*_\alpha$ is a $\mathbb{P}_\kappa$-name corresponding to the $\mathbb{P}_\alpha$-name $\underset{\sim}{a}_\alpha$.

- Then we have $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{g}$ is a $\Diamond^{\mathcal{P},extendible}_{Laver,\kappa}$-sequence" . $\dashv$

(Proposition 9)

[3] S.F., Extendible cardinals, and Laver-generic large cardinal axioms for extendibility, extended version of the article with the same title in RIMS Kôkyûroku No.2315 (2025), 62-82, preprint.

Consistency of generic Laver Diamond (3/3)

Proposition 9. Suppose that $\mathcal{P}$ is a Σ_2 transfinitely iterable class of p.o.s containing a p.o. which provably adds a new real. If κ is an extendible cardinal, then there is a $\mathbb{P} \in \mathcal{P}$ s.t.

$$\Vdash_{\mathbb{P}_\kappa} \text{“} 2^{\aleph_0} = \kappa, \ \mathcal{P}\text{-LgLCA for extendible, and } \diamondsuit^{\mathcal{P},extendible}_{Laver,2^{\aleph_0}} \text{ holds”}.$$

- The restriction “Σ_2” on the class of p.o.s $\mathcal{P}$ can be replaced by “Σ_n” $(n > 2)$ if we start from a $C^{(n')}$-extendible κ for a large enough $n' \in \mathbb{N}$. More
   - For any $n \in \mathbb{N}$ if n' is large enough relative to n, then starting from an $C^{(n')}$-extendible κ, the same construction proves

$$\text{“} \Vdash_{\mathbb{P}_\kappa} \text{“} 2^{\aleph_0} = \kappa, \text{ super-}C^{(n)}\ \mathcal{P}\text{-LgLCA for extendible, and } \diamondsuit^{+n,\mathcal{P},extendible}_{Laver,2^{\aleph_0}} \text{ holds” ”}$$

where $\diamondsuit^{+n,\mathcal{P},extendible}_{Laver,2^{\aleph_0}}$ is the “super-$C^{(n)}$” version of the gen. Laver-diamond principle $\diamondsuit^{\mathcal{P},extendible}_{Laver,2^{\aleph_0}}$.

Laver-generic Maximum (LgM)

- Most of the known "desirable axioms" can be incorporated into a single strong form of generic Laver-diamond principle:

- Let $n \in \mathbb{N}$, $\mathcal{P}$ a (two-step) iterable class of p.o.s, and LC a notion of large large cardinal κ.

$\diamondsuit^{++\ \mathcal{P},super\text{-}C^{(n)}\text{-}\mathrm{LC}}_{Laver,\kappa}$: there is $f : \kappa \to V_\kappa$ s.t., for any set a, $C^{(n)}$-cardinal $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}, \underset{\sim}{\mathbb{Q}}, \mathbb{H} \in M$, $|\,RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| = j(\kappa)$, j satisfies the closure property corresponding to LC, $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, and $j(f)(\kappa) = a$.

- A variation of the proof of Proposition 9 shows the consistency of this $++$ version of generic Laver diamond.

- $\diamondsuit^{++\ \mathcal{P},super\text{-}C^{(n)}\text{-}\mathrm{LC}}_{Laver,\kappa}$ implies both
  - the super $C^{(n)}$ $\mathcal{P}$-LgLCA for LC, and
  - $\diamondsuit^{\mathcal{P},extendible}_{Laver,\kappa}$.

An instance of LgM

- Let $\mathcal{P}$ = the class of all semi-proper p.o.s, LC = "hyperhuge" and $\kappa = 2^{\aleph_0}$. Then

$\diamondsuit^{++\ \mathcal{P},super\text{-}C^{(\infty)}\text{-}LC}_{Laver,\kappa}$ (i.e. $\diamondsuit^{++\ \mathcal{P},super\text{-}C^{(n)}\text{-}LC}_{Laver,\kappa}$ for all $n \in \mathbb{N}$) implies, besides ⓪: $\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$,

①: MM^{++}, thus also $2^{\aleph_0} = \aleph_2$, SCH, and all other consequences of MM^{++};

②: The Maximality Principle $MP(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))$;

③: The unbounded resurection axiom $UR(\mathcal{P})$ of Tsaprounis holds;

④: ② implies Viale's Absoluteness Theorem (c.f. Francesco Parente's talk);

⑤: The bedrock exists and κ (in V) is super-$C^{(\infty)}$ hyperhuge cardinal in the bedrock. The bedrock is $\leq \kappa$-ground of V;

⑥: ⑤ implies that there are class many super-$C^{(n)}$ hyperhuge cardinals for each $n \in \mathbb{N}$;

⑦: ② and ⑥ imply that each of (practically) all principles known to be consistent with set theory is a theorem in some $\leq \kappa$-ground of V.

- E.g., this is the case with Cichoń's Maximum!

An instance of LgM (2/2)

①: S.F., Ottenbreit Maschio Rodrigues, and Sakai [4] .

②: S.F. and Usuba [5] .

③: S.F. [6] .

④: S.F., Gappo and Parente [7]

⑤: S.F., and Usuba [5] .

⑥: See the extra slide .

⑦: S.F., and Usuba [5] .

[4] S.F., A. Ottenbreit Maschio Rodrigues and H. Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Mathematical Logic, Vol.60, 3-4, (2021), 495–523.

[5] S.F. and T.Usuba, On Recurrence Axioms, Annals of Pure and Applied Logic, Vol.176, (10), (2025).

[6] S.F., Maximality Principles and Resurrection Axioms in light of Laver-generic large cardinal, preprint.

[7] S.F., T. Gappo, and F. Parente, Generic Absoluteness revisited, to appear in Journal of Symbolic Logic.

Thank you for your attention!
ご清聴ありがとうございました.
Vielen Dank für die Aufmerksamkeit.
Grazie mille per l'attenzione.
Muchas gracias por la atención.

Proposition 12. Suppose $n \in \mathbb{N}$ and κ is a super- $C^{(n')}$-hyperhuge cardinal fot a sufficiently large $n' > n$. Then there are class many super- $C^{(n)}$-hyperhuge cardinals.

Proof. Let $n' > n$ be s.t. "being a super-$C^{(n)}$-hyperhuge cardinal" is absolute between ${V_\lambda}^{\mathsf{V}}$ and V for a $C^{(n')}$-cardinal λ.

- For $\mu > \kappa$, let $\lambda > \mu$ be a $C^{(n')}$-cardinal, and let j, $M \subseteq \mathsf{V}$ be s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$ ①: $j''j(\lambda) \in M$, and ②: ${V_{j(\lambda)}}^{\mathsf{V}} \prec_{\Sigma_{n'}} \mathsf{V}$.
- ${V_\lambda}^{\mathsf{V}} \models$ "κ is a $C^{(n)}$-hyperhuge cardinal" by the choice of n' and λ. Hence
  - ${V_{j(\lambda)}}^{M} \models$ "$j(\kappa)$ is a $C^{(n)}$-hyperhuge cardinal" by elementarity.

  By ①, it follows that
  - ${V_{j(\lambda)}}^{\mathsf{V}} \models$ "$j(\kappa)$ is a $C^{(n)}$-hyperhuge cardinal".
- Hence by ②, $\mathsf{V} \models$ "$j(\kappa)$ is a $C^{(n)}$-hyperhuge cardinal". Since $j(\kappa) > \mu$ for an arbitrary μ, this proves the theorem.

□ (Proposition 12)

$\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{g}$ is a $\diamondsuit^{\mathcal{P},\mathit{extendible}}_{\mathit{Laver},\kappa}$-sequence"

$\vdash$ Suppose that $\mathbb{G}_\kappa$ is a $(\mathsf{V},\mathbb{P}_\kappa)$-generic filter. Let $X \in \mathsf{V}[\mathbb{G}_\kappa]$ and $\underset{\sim}{X}$ be a $\mathbb{P}_\kappa$-name of X.

- Let $g = \underset{\sim}{g}[\mathbb{G}_\kappa]$. Then $g : \kappa \to {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa]}$ and

$$g(\alpha) = \begin{cases} \underset{\sim}{a}[\mathbb{G}_\alpha], & \text{if } f(\alpha) = \langle \underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{a}_\alpha \rangle \text{ for some } \underset{\sim}{\mathbb{Q}}_\alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

- Since f is a Laver-diamond for extendible, there are j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, (*) $V_{j(\lambda)} \in M$ and $j(f)(\kappa) = \langle \underset{\sim}{\mathbb{Q}}, \underset{\sim}{X} \rangle$ for some $\mathbb{P}_\kappa$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$".
- Let $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \underset{\sim}{\mathbb{R}}$, and let $\mathbb{H}$ be $(\mathsf{V}[\mathbb{G}_\kappa], \underset{\sim}{\mathbb{R}}[\mathbb{G}_\kappa])$-generic. Let $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \to \mathsf{V}[\mathbb{G}_\kappa * \mathbb{H}]$; $\underset{\sim}{a}[\mathbb{G}_\kappa] \;\mapsto\; j(\underset{\sim}{a})[\mathbb{G}_\kappa * \mathbb{H}]$.
  - ▷ Then, we have $\tilde{j} \supseteq j$, $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \overset{\prec}{\to}_\kappa M[\mathbb{G}_\kappa * \mathbb{H}]$. Since $\mathbb{P}_\kappa$ has the κ-cc, $j(\mathbb{P}_\kappa)$ has the $j(\kappa)$-cc. Thus (*) implies that ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{G}_\kappa * \mathbb{H}]} \in M[\mathbb{G}_\kappa][\mathbb{H}]$.
  - ▷ Also, $\tilde{j}(g)(\kappa) = j(\underset{\sim}{g})(\kappa)[\mathbb{G}_\kappa] = \underset{\sim}{X}[\mathbb{G}_\kappa] = X$.
- Thus $V[\mathbb{G}_\kappa] \models$ "g is a $\diamondsuit^{\mathcal{P},\mathit{extendible}}_{\mathit{Laver},\kappa}$-sequence".

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