Generic solution of Hamburger's Problem

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The most up-to-date version of these slides is going to be downloadable as
https://fuchino.ddo.jp/slides/generic-hamburger-pf.pdf

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to: Outline

Pictures of the blackboards at the talk are available at:
https://fuchino.ddo.jp/kobe/bbd/Scan_2024-04-24-21.47_generic_hamburger_bb.pdf

References

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[S.F.] S.F., Resurrection and Recurrence, to appear in the Festschrift on the occasion of the 75. birthday of Professor Janos Makowsky.
https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschrift-x.pdf

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Outline

The Hamburger's Question in original from

- Is the following statement consistent?
  — a question asked by P. Hamburger in 1970's

(The Hamburger's Statement in original form): For any non-metrizable topological space X, there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

- A topological space $X = (X, \mathcal{O})$ is metrizable if there is a metric d on X s.t. $\mathcal{O} = \mathcal{O}_d$ where $\mathcal{O}_d$ is the topology induced from d. X is non-metrizable if there is no such metric d.
- Note that if X is metrizable then any subspace of X is also metrizable.

(Reformulation of the Hamburger's Statement): For any topological space X, X is metrizable if and only if all subspace Y of X of cardinality $< \aleph_2$ are metrizable.

The Hamburger's statement in its original form is false

- Is the following statement consistent?
  — a question asked by P. Hamburger in 1970's

(Hamburger's Statement in original form): For any non-metrizable topological space X, there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

The statement above is false.

Example 1 (Hajnal and Juhász (1976)). For any set X of uncountable cardinality and $p \notin X$. Let $\mathcal{O}$ be the topology on $X \cup \{p\}$ generated by

$$\{\{a\} : a \in X\} \cup \{X \cup \{p\} \setminus A : A \in [X]^{<|X|}\}.$$

- $X \cup \{p\}$ is non-metrizable but any $Y \in [X \cup \{p\}]^{<|X|}$ is metrizable.

Revised Hamburger's Problem HH

- The following revised problem is called as Hamburger's Problem today:

- Is the following hypothesis consistent?

(HH): For any first countable (i.e. $\chi(X) = \aleph_0$) topological space X, X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

- This problem is still open.

- However several partial (positive) answers are known.

Known results around HH

(HH): For any first countable (i.e. $\chi(X) = \aleph_0$) topological space X, X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

Lemma 2. The assertion of **HH** with $< \aleph_2$ replaced by $< \aleph_1$ is false.

Prop. 3 (Hajnal and Juhász (1976)). If there is a non-reflecting stationary $S \subseteq E^{\lambda}_{\omega}$ for a regular cardinal λ then **HH** does not hold.

Cor. 4. **HH** implies that $\neg\Box_\kappa$ holds for all uncountable κ.

Thm. 5 (A. Dow). For any countably compact space X, the space X is non-metrizable if and only if there is a subspace Y of X of size $< \aleph_2$ s.t. Y is non-metrizable.

Thm. 6 ([Juhász-S.F.-et al]) . The following follows from MA^+(σ-closed): For any locally countably compact space X, the space X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

Known results around HH (2/3)

(HH): For any first countable (i.e. $\chi(X) = \omega$) topological space X, X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

Thm. 6 ([Juhász-S.F.-et al]). The following follows from $\mathrm{MA}^+(\sigma\text{-closed})$: For any locally countably compact space X, the space X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

- The following Proposition (an improvement of the result by Hajnal and Juhász (Prop. 3)) shows that the reflection statement in Thm. 6 is also a large cardinal property:

Prop. 7 ([Juhász-S.F.-et al]). If there is a non-reflecting stationary set $S \subseteq E^\lambda_\omega$ for a regular cardinal $\lambda > \omega_1$ then we can construct a counter-example to the reflection statement of Thm. 6.

Cor. 8. ([Juhász-S.F.-et al]). The reflection statement of Thm. 6 implies $\neg\square_\kappa$ for all uncountable κ. I.e. $\square_\kappa$ for a $\kappa \geq \omega_1 \rightarrow \neg$ claim of Thm.6.

Known results around HH (3/3)

- The following variation of Hamburger's Hypothesis is known to be consistent:

$(\mathrm{HH})_{<2^{\aleph_0}}$: For any topological space X with $\chi(X) < 2^{\aleph_0}$, X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $< 2^{\aleph_0}$.

Thm. 9. (A. Dow, F. Tall, and W. Weiss, 1990) $\mathrm{HH}_{<2^{\aleph_0}}$ is consistent under the consistency of a supercompact cardinal.

Proof. Adding supercompact many Cohen reals makes a model of $\mathrm{HH}_{<2^{\aleph_0}}$.

□ (Thm. 9)

An extra slide added after the first talk.

- In most of the cases, $\square_\kappa$ for some uncountable κ negates the reflection property in consideration.

- Actually the reflection property of Thm. 6 is known to be equivalent to the combinatorial principle called the Fodor-type reflection principle, and it can be characterized as a total negation of a weak variant of square principle ([1]).

[1] S.F., Hiroshi Sakai, Lajos Soukup and Toshimichi Usuba, More about Fodor-type Reflection Principle,
https://fuchino.ddo.jp/papers/moreFRP-x.pdf

- Some more explanations about generic large cardinals and Laver-generic large cardinals ($\rightarrow$ blackboard)
  [Two lemmas illustrating the background of generic large cardinals]

The Higher Infinite

Generic Hamburger's Problem GHH

- We can prove the consistency of the following modification of HH (under the consistency of an large large cardinal < 2-huge):

- In the following we assume that the topology of a topological space X is given by its open base $X = (X, \tau)$.

- Suppose that $\mathcal{P}$ is a class of p.o.s.

- A topological space $X = (X, \tau)$ is said to be $\mathcal{P}$-indestructibly non-metrizable if $\Vdash_{\mathbb{P}}$ " $\check{X} = (\check{X}, \check{\tau})$ is non-metrizable " holds for all $\mathbb{P} \in \mathcal{P}$.

(GHH$^{\mathcal{P}-}_{<\kappa}$): For any $\mathcal{P}$-indestructibly non-metrizable space X of character $< \kappa$, there is a non-metrizable subspace $Y \subseteq X$ of size $< \kappa$.

(GHH$^{\mathcal{P}}_{<\kappa}$): For any $\mathcal{P}$-indestructibly non-metrizable space X of character $< \kappa$, there is a $\mathcal{P}$-indestructibly non-metrizable subspace $Y \subseteq X$ of size $< \kappa$.

Generic Hamburger's Problem GHH (2/2)

- κ is $\mathcal{P}$-generically supercompact, if for any $\lambda \geq \kappa$ there is $\mathbb{P} \in \mathcal{P}$ s.t. for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.
- An inner model W (of ZFC) in V is said to be a ground if V is a set generic extension of W.
- For a set Π, $(\mathcal{P}, \Pi)$-Recurrence Axiom ($(\mathcal{P}, \Pi)$-RcA, for short) is the assertion:

($(\mathcal{P}, \Pi)$-RcA): For any $\mathbb{P} \in \mathcal{P}$ and $\vec{p} \in \Pi$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\vec{p})$" then there is a ground W in V s.t. $\vec{p} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\vec{p})$.

Thm. 10. (1) If $\kappa > \aleph_1$ is $\mathcal{P}$-generically supercompact, then $\mathrm{GHH}^{\mathcal{P}-}_{<\kappa}$ holds.

(2) If $\kappa > \aleph_1$ is $\mathcal{P}$-generically supercompact, and $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA holds, then $\mathrm{GHH}^{\mathcal{P}}_{<\kappa}$ holds.

Three typical instances of Thm. 10

- $\kappa_{\mathfrak{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$
- κ is (tightly) $\mathcal{P}$-Laver-gen. supercompact (superhuge) if ...

κ is $\mathcal{P}$-Laver-gen. supercompact + $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA for $\mathcal{P}$= σ-closed p.o.s. $\triangleright$ This implies $\kappa = 2^{\aleph_0} = \aleph_1$, MA^{++}($\sigma$-closed), and $\mathrm{GHH}^{\mathcal{P}}_{\leq 2^{\aleph_0}}$. (i.e. $\mathrm{GHH}^{\mathcal{P}}_{<\kappa_{\mathfrak{refl}}}$)

κ is $\mathcal{P}$-Laver-gen. supercompact + $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA for $\mathcal{P}$=semi-proper p.o.s. (or proper) $\triangleright$ This implies $\kappa = 2^{\aleph_0} = \aleph_2$, MM^{++} (or PFA^{++}), and $\mathrm{GHH}^{\mathcal{P}}_{<\aleph_2}$. (i.e. $\mathrm{GHH}^{\mathcal{P}}_{<\kappa_{\mathfrak{refl}}}$)

κ is tightly $\mathcal{P}$-Laver-gen. superhuge + $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA for $\mathcal{P}$=c.c.c. p.o.s. $\triangleright$ This implies that $\kappa = 2^{\aleph_0}$ is extremely large, a strong variant of MA, and $\mathrm{GHH}^{\mathcal{P}}_{<2^{\aleph_0}}$. (i.e. $\mathrm{GHH}^{\mathcal{P}}_{<\kappa_{\mathfrak{refl}}}$)

- In the next talk, we shall improve the axiomatic framework of these instances (in particular, replacing the assumptions with axioms whose exact consistency strength is known).

Proof of Thm. 10

- The proof will be given on the blackboards!!!

  - κ is $\mathcal{P}$-generically supercompact, if for any $\lambda \geq \kappa$ there is $\mathbb{P} \in \mathcal{P}$ s.t. for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

- An inner model W (of ZFC) in V is said to be a ground if V is a set generic extension of W.

- For a set Π, $(\mathcal{P}, \Pi)$-Recurrence Axiom ($(\mathcal{P}, \Pi)$-RcA, for short) is the assertion:

$((\mathcal{P}, \Pi)$-RcA): For any $\mathbb{P} \in \mathcal{P}$ and $\vec{p} \in \Pi$, if $\Vdash_{\mathbb{P}}$ " $\varphi(\vec{p})$ " then there is a ground W in V s.t. $\vec{p} \in$ W and W $\models \varphi(\vec{p})$.

Thm. 10. (1) If $\kappa > \aleph_1$ is $\mathcal{P}$-generically supercompact, then $\mathsf{GHH}^{\mathcal{P}-}_{<\kappa}$ holds.

(2) If $\kappa > \aleph_1$ is $\mathcal{P}$-generically supercompact, and $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA holds, then $\mathsf{GHH}^{\mathcal{P}}_{<\kappa}$ holds.

Sufficiently almost-free algebras are free

Prop. 11. ([2]) Let $\mathcal{V}$ be a universal algebraic variety and $A \in \mathcal{V}$. If $\mathbb{P}$ is a ccc p.o. and $\Vdash_{\mathbb{P}}$ "A is free" then A is really free.

[2] S.F., On Potential embedding and versions of Martin's axiom, Notre Dame Journal of Formal Logic, Vol.33.No.4 (1992), 481–492.

Thm. 12. Suppose that κ is ccc-generic supercompact. Let $\mathcal{V}$ be as above. Then For any $A \in \mathcal{V}$ of cardinality $\geq \kappa$, A is free if and only if $\{B : B \leq A, |B| < \kappa, B \text{ is free}\}$ contains a club in $[A]^{<\kappa}$.

Cor. 13. Suppose that κ is ccc-generic supercompact. Then any (abelian) group G is not a(n abelian) free group if there is $H \leq G$ of cardinality $< \kappa$ s.t. H is non-free.

Proof. Any subgroup of (abelian) free group is (abelian) free. [Nielsen-Schreier Theorem (Dedekind Theorem)] ◻ (Cor. 13)

Cor. 14. Suppose that κ is ccc-generic supercompact. Then any Boolean algebra B is non-free if and only if there are stationary many non-free $A \leq B$ of cardinality $< \kappa$. ◻

Restricted Recurrence Axioms

- For $n \in \omega$,

($(\mathcal{P}, \Pi)_{\Sigma_n}$-RcA): For any $\mathbb{P} \in \mathcal{P}$, $\vec{p} \in \Pi$, and Σ_n-formula φ, if $\Vdash_{\mathbb{P}}$ "$\varphi(\vec{p})$" then there is a ground W in V s.t. $\vec{p} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\vec{p})$.

Lemma 15. "X is metrizable" is Σ_1: over the parameters $\langle X, \tau, \mathbb{Q}\rangle$, $(\exists d \ \cdots)$

Thm. 16 ([S.F.&Usuba]). Suppose that κ is tightly $\mathcal{P}$-Lever-gen. ultrahuge for an iterable class $\mathcal{P}$ of p.o.s. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-RcA holds (the theorem in [S.F.& Usuba] is slightly stronger than this statement).

Cor. 17. Suppose that κ is tightly $\mathcal{P}$-Lever-gen. ultrahuge for one of the "typical" classes $\mathcal{P}$ of p.o.s on the previous slide. Then $\mathsf{GHH}^{\mathcal{P}}_{\kappa_{\mathfrak{refl}}}$ holds.

Proof. Just repeat the proof of Thm. 10, (1) using Lemma 15 and Thm. 16. □ (Cor. 17)

Generic Reflection Principle

- For a topological structure $\mathfrak{A}$ (like topological group, Banach algebra, etc. but also purely algebraic structure without topology like group, abelian group, Boolean algebra, etc.) a property P of $\mathfrak{A}$ is $\mathcal{P}$-indestructible (or $\mathfrak{A}$ is $\mathcal{P}$-indestructibly P) if $\Vdash_{\mathbb{P}}$ " $\mathfrak{A}$ has the property P " for all $\mathbb{P} \in \mathcal{P}$.

($\mathrm{GR}^{\mathcal{P}}_{<\kappa}$): Suppose that $\mathcal{T}$ is any class of topological structures s.t. all $\mathfrak{A} \in \mathcal{T}$ is of character $<\kappa$ (as topological space). Then, for any Σ_1 property P of elements of $\mathcal{T}$, if $\mathfrak{A} \in \mathcal{T}$ is $\mathcal{P}$-indestructibly $\neg P$, there are stationarily many (topological) substructures $\mathfrak{B}$ of $\mathfrak{A}$ of size $<\kappa$ which is also $\mathcal{P}$-indestructibly $\neg P$.

Cor. 17A. Suppose that κ is tightly $\mathcal{P}$-Lever-gen. ultrahuge for one of the "typical" classes $\mathcal{P}$ of p.o.s on the previous slide. Then $\mathrm{GR}^{\mathcal{P}}_{\kappa_{\mathfrak{refl}}}$ holds.

Proof. Similarly to Cor. 17. □ (Cor. 17A)

- Note that reflection statements of Thm.10., Thm.12 are special cases of Cor.17A

Laver-genericity with exact consistency strength

- An inner model $\overline{\mathsf{W}}$ of V is said to be a bedrock if it is the minimal ground.

Thm. 18 ([S.F.&Usuba]). Suppose that κ is tightly $\mathcal{P}$-gen. hyperhuge (this is stronger than tightly $\mathcal{P}$-gen. ultrahuge), then there is a bedrock and κ is a hyperhuge cardinal in the bedrock. □

Cor. 19 ([S.F.&Usuba]). For a natural class $\mathcal{P}$ (as in the three instances of Thm. 10) of p.o.s the following statements are equiconsistent ((a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are implications):

(a) $\kappa = \kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-gen. hyperhuge.
(b) There is a κ which is tightly $\mathcal{P}$-gen. hyperhuge.
(c) $\kappa_{\mathfrak{refl}}$ is hyperhuge in the bedrock of V.
(d) There is a hyperhuge cardinal.

Laver-genericity with exact consistency strength (2/3)

- For most of the notions of large large cardinals, The tightly $\mathcal{P}$-Laver-gen. large cardinal axiom (the axiom asserting the existence of such a cardinal) does not imply the full $(\mathcal{P}, \emptyset)$-RcA.

— Theorem 5.11 in [S.F. ∞]

[S.F. ∞] S.F., Maximality Principles and Resurrection Axioms under a Laver-generic large cardinal, preprint,
https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf

Thm. 20 ([S.F.&Usuba]). Suppose that $\mathcal{P}$ is iterable class of p.o.s and κ is tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-gem. ultrahuge cardinal. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA holds. ↑ definition: on the blackboard →

Laver-genericity with exact consistency strength (3/3)

Thm. 20 ([S.F.&Usuba]). Suppose that $\mathcal{P}$ is iterable class of p.o.s and κ is tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-gem. ultrahuge cardinal. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA holds. ↑ definition: on the blackboard →

Proof. Similarly to the proof of Thm. 16. ⧉ (Thm. 20)

Thm. 21 ([S.F.&Usuba]). Let $\mathcal{P}$ be as above. Then the following statements are equiconsistent. Their consistency follows from a 2-huge cardinal:

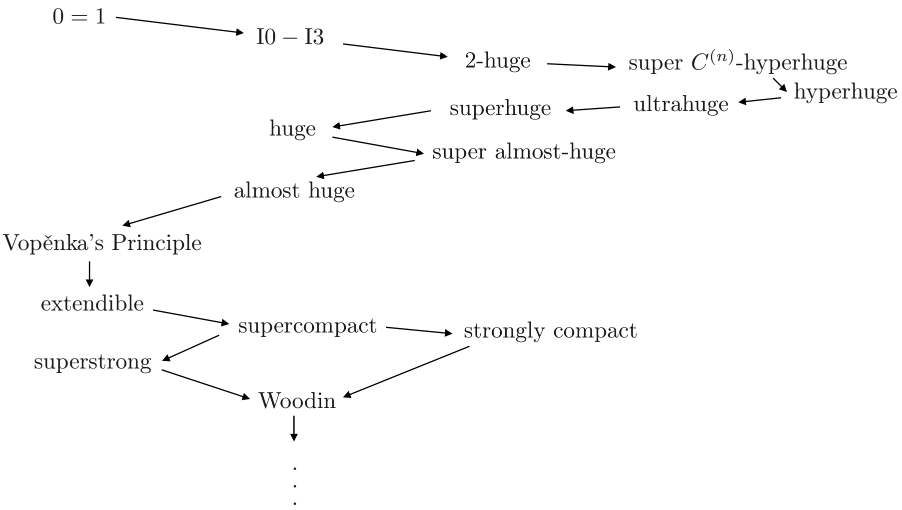
(a) $\kappa = \kappa_{\mathfrak{refl}}$ is tightly super $C^{(\infty)}$- $\mathcal{P}$-Laver-gen. hyperhuge cardinal.

(b) $\kappa_{\mathfrak{refl}}$ is super $C^{(\infty)}$-hyperhuge in the bedrock of V.

Proof. By (the proof of) Thm. 18. ⧉ (Thm. 22)

Open problem. Can "hyperhuge" in Thm. 18 (and Cor. 19, Thm. 21) be replaced by "ultrahuge" (or by "superhuge" or even by "super almost-huge") ? — for genuine large cardinals "hyperhuge" can be replaced by "extendible" by a result of Usuba.

The Higher Infinite



Thank you for your attention!

Grazie per l'attenzione!

Two Lemmas in connection with generic large cardinals

Lemma A1. (Proposition 22.4 in [higher-infinite]) Suppose that U is an ω_1-complete ultrafilter over a set S and $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ is the elementary embedding induced from U then for any cardinal γ, $j''\gamma \in M$ if and only if ${}^\gamma M \subseteq M$.

Lemma A2. (Lemma 2.5 in S.F., Rodrigues and Sakai [II]) Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter for a p.o. $\mathbb{P} \in \mathsf{V}$, and $j : \mathsf{V} \overset{\prec}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$ is s.t., for cardinals κ, λ in V with $\kappa \leq \lambda$, $crit(j) = \kappa$ and $j''\lambda \in M$.

(1) For any set $A \in \mathsf{V}$ with $\mathsf{V} \models |A| \leq \lambda$, we have $j''A \in M$.

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$.

(3) For any $A \in \mathsf{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.

(4) $(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, Thus, if $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.

(6) $j \restriction A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.