On Magidor's characterization of supercompact cardinals as Löwenheim-Skolem numbers of the second-order logic

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https://fuchino.ddo.jp/index.html

(2022 年 06 月 04 日 (18:01 JST) printer version)

2022 年 5 月 25 日 (16:00 〜 JST), 至 Kobe Set Theory Seminar

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The subject of the talk is related to the research supported by Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717

Downward Löwenheim-Skolem Theorem for First-Order Logic

- **Notation:** A structure $\mathfrak{A}$ is a (first-order) structure of countable signature (if not mentioned otherwise).

- For a structure $\mathfrak{A}$, we denote with $|\mathfrak{A}|$ the underlying set of $\mathfrak{A}$, and $\|\mathfrak{A}\|$ the cardinality (of the underlying set) of $\mathfrak{A}$.

  Cf.: if X is a set, we denote with $|X|$ the cardinality of X.

Theorem 1. (Downward Löwenheim-Skolem Theorem) For any uncountable cardinal κ and a structure $\mathfrak{A}$ (of countable signature) if $S \subseteq |\mathfrak{A}|$ is of cardinality $< \kappa$, then there is $\mathfrak{B} \prec \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa$.

Löwenheim-Skolem Spectrum of a Logic

- Let $\mathcal{L}$ be a logic with a notion $\prec_{\mathcal{L}}$ of elementary substructure. The Löwenheim-Skolem spectrum of the logic $\mathcal{L}$ is defined as:

$$\mathrm{LSS}(\mathcal{L}) := \{\mu \in \mathrm{Card} : \text{for any structure } \mathfrak{A} \text{ of a countable signature and } S \subseteq |\mathfrak{A}| \text{ with } |S| < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } S \subseteq |\mathfrak{B}| \text{ and } \|\mathfrak{B}\| < \mu\}.$$

- Denoting the first-order logic with L, (the classical) Downward Löwenheim-Skolem Theorem can be reformulated as:

Theorem 2. $\mathrm{LSS}(L) = \{\kappa \in \mathrm{Card} : \kappa \geq \aleph_1\}$.

Lemma 2a. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathrm{LSS}(\mathcal{L}) = \{\mu \in \mathrm{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof

LSS($\mathcal{L}$) is closed

Lemma 2b. For any logic $\mathcal{L}$, LSS($\mathcal{L}$) is a closed class of cardinals.

Proof. Suppose that $\langle \kappa_\alpha : \alpha < \delta \rangle$ is a strictly increasing sequence in LSS($\mathcal{L}$) and $\kappa = \sup_{\alpha<\delta} \kappa_\alpha$. We want to show that $\kappa \in$ LSS($\mathcal{L}$).

- Suppose that $\mathfrak{A}$ is a structure and $S \subseteq [\,|\mathfrak{A}|\,]^{<\kappa}$. Let $\alpha < \delta$ be s.t. $|\,S\,| < \kappa_\alpha$. Since $\kappa_\alpha \in$ LSS($\mathcal{L}$), there is a $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa_\alpha < \kappa$. This shows that $\kappa \in$ LSS($\mathcal{L}$). ◻ (Lemma 2b)

Löwenheim-Skolem Spectrum of some non-first-order logics

- Let $L(Q)$ be the logic obtained from the first-order logic by adding a new unary (first-order) quantifier Q.
  Interpretation: Qx ... $\Leftrightarrow$ "there are uncountably many x s.t. ...".
- The proof of the following theorem was given in my previous talk at Tokyo Model Theory Seminar (see the [slides of the talk]):

Theorem 3. $\mathrm{LSS}(L(Q)) = \{\kappa \in \mathrm{Card} : \kappa \geq \aleph_2\}$.

- $\mathcal{L}^{\aleph_0}_{stat}$ is the monadic second order logic whose second-order variables run over countable subsets of the underlying set of the structure, with new quantifier with the quantification $stat\ \underbrace{X}_{\text{second-order variable}}$ whose interpretation is "there are stationarily many X s.t. ..."
  In my next talk, I will present some results about $\mathrm{LSS}(\mathcal{L}^{\aleph_0}_{stat})$. E.g.:

Theorem 4. (see Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [2021])

For any $n \in \mathbb{N}$, $n \geq 2$, the statement "$\aleph_n = \min \mathrm{LSS}(\mathcal{L}^{\aleph_0}_{stat})$" is independent from ZFC (modulo a large cardinal).

Full second order logic

- $\mathcal{L}^{\mathrm{II}}$ denotes the (monadic, full) second-order logic with second-order variables X, Y, Z etc. running over all subsets of the underlying set of a structure. In addition to the constructs of the first-order logic, we have the symbol ε as a logical binary predicate and allow the expression "$x \mathrel{\varepsilon} X$" for a first order variable x and a second-order variable X as an atomic formula. We also allow the quantification of the form "$\exists X$" (and its dual "$\forall X$") over the second-order variables X.
- The relation symbol ε is interpreted as the (real) element relation and the interpretation of the quantifier $\exists X$ in $\mathcal{L}^{\mathrm{II}}$ is defined by:

$$\mathfrak{A} \models \exists X \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, X) \;:\Leftrightarrow$$

there exists a $B \in \mathcal{P}(\,|\mathfrak{A}|\,)$ s.t. $\mathfrak{A} \models \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, B)$

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\mathrm{II}}$-formula φ in the signature of the structure $\mathfrak{A}$ with $\varphi = \varphi(x_0, \ldots, x_{m-1}, X_0, \ldots, X_{n-1}, X)$ where $x_0, \ldots, x_{m-1}$ and $X_0, \ldots, X_{n-1}$, X are first- and second-order variables, $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$, and $B_0, \ldots, B_{n-1} \in \mathcal{P}(\,|\mathfrak{A}|\,)$.

Full second order logic (2/6)

$\mathfrak{B} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A} \;:\Leftrightarrow\; \mathfrak{B} \models \varphi(b_0, \ldots, b_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{n-1})$ holds for all formulas $\varphi = \varphi(x_0, \ldots)$ in $\mathcal{L}^{\mathrm{II}}$ without free second-order variables, and for all $b_0, \ldots, b_{n-1} \in |\mathfrak{B}|$.

▷ Exclusion of second-order free variables and parameters in this context is natural because of the following trivial example:

Example 5. Let $\mathfrak{B} \subsetneq \mathfrak{A}$. Let $B = |\mathfrak{B}|$. Then

$$\mathfrak{A} \models \exists x\,(x \not\varepsilon B) \qquad \text{but} \qquad \mathfrak{B} \models \neg\exists x\,(x \not\varepsilon B).$$

Theorem 6. (M. Magidor [1971])

$\mathrm{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa \;:\; \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

▶ A cardinal κ is supercompact if, for any $\lambda \geq \kappa$, there are transitive class M and elementary embedding $j : \mathsf{V} \to M$ s.t. κ is the smallest ordinal moved by j (critical point of j: we denote these conditions as $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$), $j(\kappa) > \lambda$ and $[M]^\lambda \subseteq M$.

Back to the proof of Proposition 12.

Full second order logic (3/6)

Theorem 6. (M. Magidor [1971])
$\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

Proof. "$\supseteq$": Since $\mathsf{LSS}(\mathcal{L}^{\mathrm{II}})$ is closed (Lemma 2b), it is enough to prove that supercompact cardinals belong to $\mathsf{LSS}(\mathcal{L}^{\mathrm{II}})$.

- ▶ Suppose that κ is supercompact and $\mathfrak{A}$ a structure in a countable signature. W.l.o.g., $|\mathfrak{A}|$ is a cardinal $\lambda_0 < \lambda$ and let $S \subseteq [\lambda_0]^{<\kappa}$ $(= [\ |\mathfrak{A}|\ ]^{<\kappa})$
- ▶ Let $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$ be s.t. $j(\kappa) > \lambda$ and $[M]^\lambda \subseteq M$.
- ▷ Then $\mathfrak{A}$, $j(\mathfrak{A}) \restriction j''\lambda_0$, $j \restriction \lambda_0 \in M$, $M \models j \restriction \lambda_0 : \mathfrak{A} \overset{\cong}{\rightarrow} j(\mathfrak{A}) \restriction j''\lambda_0$ and $\mathcal{P}(\,|\mathfrak{A}|\,)^{\mathsf{V}} = \mathcal{P}(\,|\mathfrak{A}|\,)^M$. For any $\mathcal{L}^{\mathrm{II}}$-formula $\varphi = \varphi(x_0, \ldots)$ without free second order variables, and any $a_0, \ldots \in |\mathfrak{A}|$,

$$M \models j(\mathfrak{A}) \models \varphi(j(a_0), \ldots) \quad \Leftrightarrow \quad \mathsf{V} \models \mathfrak{A} \models \varphi(a_0, \ldots)$$
$$\Leftrightarrow \quad M \models \mathfrak{A} \models \varphi(a_0, \ldots) \quad \Leftrightarrow \quad M \models j(\mathfrak{A}) \restriction j''\lambda_0 \models \varphi(j(a_0), \ldots,).$$

- ▶ Thus $M \models j(\mathfrak{A}) \restriction j''\lambda_0 \prec_{\mathcal{L}^{\mathrm{II}}} j(\mathfrak{A})$,

$$\|j(\mathfrak{A}) \restriction j''\lambda_0\| < j(\kappa),\ j(S) = j''S \subseteq |j(\mathfrak{A}) \restriction j''\lambda_0|\,.$$

- ▷ By elementarity, $\mathsf{V} \models$ there is $\mathfrak{B} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa$.

Full second order logic (4/6)

Theorem 6. (M. Magidor [1971])
LSS($\mathcal{L}^{\rm II}$) $= \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

"$\subseteq$": The proof of this direction uses the following characterization of supercompact cardinals by Magidor:

Theorem 7. (M. Magidor [1971], see Theorem 22.10 [Kanamori])

A cardinal κ is supercompact

$\Leftrightarrow$ for class many $\zeta > \kappa$, there is $\alpha < \kappa$ with $e : V_\alpha \xrightarrow{\prec}_\delta V_{\zeta+\omega}$ for a $\delta < \alpha$ s.t. $e(\delta) = \kappa$.

Back to p.11

Full second order logic (5/6)

Theorem 6. (M. Magidor [1971])
$\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

"$\subseteq$": Assume that $\kappa \in \mathsf{LSS}(\mathcal{L}^{\mathrm{II}})$ and suppose $\mu < \kappa$. We have to show that there is a supercompact cardinal δ with $\mu < \delta \leq \kappa$.

- First, note that there is an $\mathcal{L}^{\mathrm{II}}$-sentence φ^* s.t.

▷ $\langle X, E \rangle \models \varphi^* \Leftrightarrow E$ is well-founded and extensional binary relation and $mcol(\langle X, E \rangle) = \langle V_\gamma, \in \rangle$ for some γ.

For each $\lambda \geq \kappa$, let $\mathfrak{A}_\lambda = \langle V_{\lambda+\omega}, \kappa, \in \rangle$. By the choice of κ, there is $\mathfrak{B}_{\mu,\lambda} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A}_\lambda$ s.t. (1) $\mu \subseteq |\mathfrak{B}_{\mu,\lambda}|$ and (2) $\|\mathfrak{B}_{\mu,\lambda}\| < \kappa$.

Full second order logic (6/6)

Theorem 6. (M. Magidor [1971])

LSS($\mathcal{L}^{\mathrm{II}}$) $= \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

"$\subseteq$": Assume that $\kappa \in$ LSS($\mathcal{L}^{\mathrm{II}}$) and suppose $\mu < \kappa$. We have to show that there is a supercompact cardinal δ with $\mu < \delta \leq \kappa$.

- First, note that there is an $\mathcal{L}^{\mathrm{II}}$-sentence φ^* s.t.
  - $\langle X, E\rangle \models \varphi^* \Leftrightarrow E$ is well-founded and extension binary relation and $mcol(\langle X, E\rangle) = \langle V_\gamma, \in\rangle$ for some γ.

For each $\lambda \geq \kappa$, let $\mathfrak{A}_\lambda = \langle V_{\lambda+\omega}, \kappa, \in\rangle$. By the choice of κ, there is $\mathfrak{B}_{\mu,\lambda} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A}_\lambda$ s.t. (1) $\mu \subseteq |\mathfrak{B}_{\mu,\lambda}|$ and (2) $\|\mathfrak{B}_{\mu,\lambda}\| < \kappa$.

- We have $\mathfrak{B}_{\mu,\lambda} \models \varphi^*$ by elementarity and since $\mathfrak{A}_\lambda \models \varphi^*$. Hence the Mostowski collapse of $\mathfrak{B}_{\mu,\lambda}$ is of the form $\langle V_\beta, \delta, \in\rangle$. Let $e_{\mu,\lambda} : V_\beta \overset{\cong}{\to} \mathfrak{B}_{\mu,\lambda} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A}_{\mu,\lambda}$ be the inverse of Mostowski collapsing function.
  - Then we have $e_{\mu,\lambda} \restriction \mu = id_\mu$ by (1). Hence the critical point $\delta_{\mu,\lambda}$ of $e_{\mu,\lambda}$ is somewhere between μ and κ (i.e. $\mu \leq \delta_{\mu,\lambda} \leq \kappa$).
  - Since there are only set many such cardinals, there is $\mu \leq \delta^*_\mu \leq \kappa$ s.t. there are class many λ's s.t. $\delta_{\mu,\lambda} = \delta^*_\mu$.
- By Theorem 7 , it follows that δ^*_μ is supercompact. □ (Theorem 6)

A slight modification of Magidor's theorem

Theorem 6. (M. Magidor [1971])
$\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

- The proof of "$\supseteq$" of Theorem 6. actually shows the following:
  - Let $\mathcal{L}^{\mathrm{HO}}$ denote the higher order logic that is the union of nth order logics for all $n \in \omega$.
  - Note that, if $\mathcal{L}'$ has more expressive power than $\mathcal{L}$ then we have $\mathsf{LSS}(\mathcal{L}') \subseteq \mathsf{LSS}(\mathcal{L})$.

Corollary 8. (M. Magidor [1971])
$\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \mathsf{LSS}(\mathcal{L}^{\mathrm{HO}})$
$= \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

Proof. $\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) \underbrace{\supseteq}_{\text{by the remark above}} \mathsf{LSS}(\mathcal{L}^{\mathrm{HO}}) \underbrace{\supseteq}_{\text{by a modification of the proof of Theorem 6. "}\supseteq\text{"}} \{\kappa : \cdots\}$

$\underbrace{\supseteq}_{\text{Theorem 6. "}\subseteq\text{"}} \mathsf{LSS}(\mathcal{L}^{\mathrm{II}})$

□ (Corollary 8)

The spectrum of compactness numbers of a logic

- For a logic $\mathcal{L}$, the compactness spectrum of $\mathcal{L}$ is defined as:

$\mathrm{CS}(\mathcal{L}) := \{\kappa \in \mathrm{Card}$: for any $\mathcal{L}$-theory T (possibly of an uncountable signature), of size κ, T is satisfiable if and only if all $S \in [T]^{<\kappa}$ are satisfiable$\}$.

- The strong compactness number of a logic $\mathcal{L}$ is defined as:

$\mathrm{scn}(\mathcal{L}) := \min(\{\kappa \in \mathrm{Card}$: for any $\mathcal{L}$-theory T (possibly of an uncountable signature) of any size, T is satisfiable if and only if all $S \in [T]^{<\kappa}$ are satisfiable$)\}$.

Lemma 9. For a logic $\mathcal{L}$, $\{\kappa \in \mathrm{Card} : \mathrm{scn}(\mathcal{L}) \leq \kappa\} \subseteq \mathrm{CS}(\mathcal{L})$.

Proposition 10.

$\mathrm{scn}(\mathcal{L}^{\mathrm{II}}) \leq$ the smallest extendible cardinal.

The spectrum of compactness numbers of a logic (2/3)

Proposition 10. (follows from Theorem 11 below.)

$\mathrm{scn}(\mathcal{L}^{\mathrm{II}}) \leq$ the smallest extendible cardinal.

- A cardinal κ is extendible if, for any $\eta > 0$, there is $j : V_{\kappa+\eta} \xrightarrow{\preceq}_{\kappa} V_{\zeta}$ for some ζ with $\eta < j(\kappa)$.

- For a cardinal κ, $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$ is the logic defined like $\mathcal{L}^{\mathrm{II}}$ but also conjunction and disjunction of $< \kappa$ many formulas are allowed (while the number of free variables in such formulas is always kept finite).

Theorem 11. (M. Magidor [1971]), see Theorem 23.4 in [Kanamori]

The following are equivalent for $\kappa > \omega$:

(a) κ is extendible.

(b) for any $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$-theory T^*, if all $T \in [T^*]^{<\kappa}$ are satisfiable, then T^* is also satisfiable.

- κ is the least extendible cardinal $\underbrace{\Rightarrow}_{\text{by Theorem 11}}$ $\kappa = \mathrm{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) \geq \mathrm{scn}(\mathcal{L}^{\mathrm{II}})$.

The spectrum of compactness numbers of a logic (3/3)

I will go into more detail of the following theorems in my next talk on Jun 1.:

Theorem?? 12. (M. Magidor) Let κ be the least extendible cardinal. Then $\mathsf{scn}(\mathcal{L}^{\rm II}) = \mathsf{scn}(\mathcal{L}^{\rm II}_{\kappa,\omega}) = \kappa$.

Theorem 13. If κ is σ-closed-gen. supercompact, then $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0}_{stat})$.

Theorem?? 14.
If κ is σ-closed-gen. super-almost-huge, then $\mathsf{scn}(\mathcal{L}^{\aleph_0}_{stat}) \leq \kappa$.

(Theorem 14 is false in this form: for a correct version of the theorem see the slides of the next talk).

- Note that σ-closed-gen. supercompact/super-almost-huge cardinals can be "small". For example, $\aleph_n$ for any $n \geq \aleph_2$ can be σ-closed-gen. supercompact/super-almost-huge.

Generically large cardinals

- For a class $\mathcal{P}$ of p.o.s,

A cardinal κ is generically supercompact by $\mathcal{P}$ ($\mathcal{P}$ gen. supercompact, for short) if, for any $\lambda \geq \kappa$, there is $\mathbb{P} \in \mathcal{P}$ s.t., for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

A cardinal κ is generically super-almost-huge by $\mathcal{P}$ ($\mathcal{P}$ gen. superhuge, for short) if, for any $\lambda \geq \kappa$, there is $\mathbb{P} \in \mathcal{P}$ s.t., for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$ and $j''\mu \in M$ for all $\mu < j(\kappa)$.

Theorem 13. If κ is σ-closed-gen. supercompact, then $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0}_{stat})$.

Theorem?? 14.
If κ is σ-closed-gen. super-almost-huge, then $\mathsf{scn}(\mathcal{L}^{\aleph_0}_{stat}) \leq \kappa$.

(Theorem 14 is false in this form: for a correct version of the theorem see the slides of the next talk).

Thank you for your attention!
ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다

Gracias por su atención.

Dziękuję za uwagę.

Grazie per l'attenzione.

Dank u voor uw aandacht.

Ich danke Ihnen für Ihre Aufmerksamkeit.

On the restriction to countable signatures

Lemma 2a. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathsf{LSS}(\mathcal{L}) = \{\mu \in \mathsf{Card} \,:\, \text{for any structure } \mathfrak{A} \text{ with a signature of size} < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathsf{LSS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha}\rangle_{n\in\omega,\alpha<\nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.

▷ Let $R_n := \bigcup_{\alpha<\nu}\{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n\rangle_{n\in\omega}$. Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct a submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with the same underlying set and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

On the restriction to countable signatures (2/2)

Lemma 2a. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathrm{LSS}(\mathcal{L}) = \{\mu \in \mathrm{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathrm{LSS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha}\rangle_{n\in\omega,\alpha<\nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.
Let $R_n := \bigcup_{\alpha<\nu}\{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n\rangle_{n\in\omega}$. Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct an $\mathcal{L}$-elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with the same underlying set.

"$\supseteq$": Suppose now that μ is in the set on the right side of the equality. Let $\mathfrak{A}$ be a structure of size $\geq \mu$ with a countable signature, and $S \in [\,|\mathfrak{A}|\,]^{<\mu}$.
Let $\mathfrak{A}^+ = \langle \mathfrak{A}, a\rangle_{a\in S}$. Applying the assumption on μ, we obtain $\mathfrak{B}^+ \prec_{\mathcal{L}} \mathfrak{A}^+$ of size $< \mu$. Denoting by $\mathfrak{B}$ the $\mathfrak{B}^+$ reduced to the original language, we have $\|\mathfrak{B}\| < \mu$, $S \subseteq |\mathfrak{B}|$ and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

⧉ (Lemma 2a)

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