

On Löwenheim-Skolem number and compactness number of some non first-order logics

slides of the last talk

Sakaé Fuchino (渚野 昌)

Kobe University, Japan

<https://fuchino.ddo.jp/index.html>

(2022年06月19日 (17:49 JST) printer version)

2022年6月1日 (16:00 ~ JST), 至 **Kobe Set Theory Seminar**

The following slides are typeset by up \LaTeX with beamer class, and presented on UP2 Version 2.0.0 by Ayumu Inoue running on an ipad pro (10.5inch).

The most up-to-date version of these slides is going to be downloadable as

<https://fuchino.ddo.jp/slides/kobe-fuchino-2022-06-01-pf.pdf>

The subject of the talk is related to the research supported by Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717

- ▶ Löwenheim-Skolem Spectrum of a Logic
- ▶ Stationary logic with two notions of elementary substructure
- ▶ Full second order logic
- ▶ A slight modification of Magidor's theorem
- ▶ The spectrum of compactness numbers of a logic
- ▶ Strong compactness number of the infinitary second-order logics
- ▶ Strong compactness number of the full second-order logic
- ▶ Examples showing non-trivial situations
- ▶ Generic large cardinals
- ▶ Strong compactness number of stationary logic
- ▶ Generic view of the strong compactness number of stationary logic

- ▶ **Notation:** A structure \mathfrak{A} is a (first-order) structure of countable signature (if not mentioned otherwise).
- ▷ For a structure \mathfrak{A} , we denote with $|\mathfrak{A}|$ the underlying set of \mathfrak{A} , and $\|\mathfrak{A}\|$ the cardinality (of the underlying set) of \mathfrak{A} .
Cf.: if X is a set, we denote with $|X|$ the cardinality of X .
- ▶ Let \mathcal{L} be a logic with a notion $\prec_{\mathcal{L}}$ of elementary substructure. The Löwenheim-Skolem spectrum of the logic \mathcal{L} is defined as:

$$\text{LSS}(\mathcal{L}) := \{\mu \in \text{Card} : \text{for any structure } \mathfrak{A} \text{ of a countable signature} \\ \text{and } S \subseteq |\mathfrak{A}| \text{ with } |S| < \mu, \\ \text{there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } S \subseteq |\mathfrak{B}| \text{ and } \|\mathfrak{B}\| < \mu\}.$$

$LSS(\mathcal{L}) := \{\mu \in \text{Card} : \text{for any structure } \mathfrak{A} \text{ of a countable signature}$
and $S \subseteq |\mathfrak{A}|$ with $|S| < \mu$,
there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \mu\}$.

Lemma 1. For a logic \mathcal{L} (with natural properties expected to a “logic”), we have

$LSS(\mathcal{L}) = \{\mu \in \text{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of}$
size $< \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $\|\mathfrak{B}\| < \mu\}$.

Proof

Lemma 2. For any logic \mathcal{L} , $LSS(\mathcal{L})$ is a closed class of cardinals.

Proof

- $\mathcal{L}_{stat}^{\aleph_0}$ is the monadic second order logic whose second-order variables run over countable subsets of the underlying set of the structure, with new quantifier with the quantification $\underbrace{stat X}$ whose interpretation is “there are stationarily many X s.t. ...” second-order variable

$\mathfrak{B} \prec_{\mathcal{L}_{stat}^{\aleph_0}} \mathfrak{A} :\Leftrightarrow \mathfrak{B} \models \varphi(b_0, \dots, b_{m-1}, A_0, \dots, A_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \dots, b_{m-1}, A_0, \dots, A_{n-1})$ holds for all $\mathcal{L}_{stat}^{\aleph_0}$ -formulas $\varphi = \varphi(x_0, \dots, x_0, \dots)$, for all $b_0, \dots, b_{m-1} \in |\mathfrak{B}|$, and for all $A_0, \dots, A_{n-1} \in [|\mathfrak{B}|]^{\aleph_0}$.

$\mathfrak{B} \prec_{\mathcal{L}_{stat}^{\aleph_0}}^- \mathfrak{A} :\Leftrightarrow \mathfrak{B} \models \varphi(b_0, \dots, b_{m-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \dots, b_{m-1})$ holds for all $\mathcal{L}_{stat}^{\aleph_0}$ -formulas $\varphi = \varphi(x_0, \dots, x_{m-1})$ without free second-order variables, and for all $b_0, \dots, b_{m-1} \in |\mathfrak{B}|$.

- In the following, we consider the logic $\mathcal{L}_{stat}^{\aleph_0}$ as equipped with $\prec_{\mathcal{L}_{stat}^{\aleph_0}}$. The logic $\mathcal{L}_{stat}^{\aleph_0}$ equipped with $\prec_{\mathcal{L}_{stat}^{\aleph_0}}^-$ is denoted with $\mathcal{L}_{stat}^{\aleph_0 -}$.

Stationary logic with two notions of elementary substructure (2/2) Lösko & compactness no. (6/24)

- ▶ $\mathcal{L}_{stat}^{\aleph_0, II}$ is like $\mathcal{L}_{stat}^{\aleph_0}$ but the quantification $\exists X$ and $\forall X$ of the second order variables X is allowed in addition. These quantifiers are then interpreted as: "there exists a countable subset X of the underlying set of the structure s.t. ..." and "for all countable subsets X of the underlying set of the structure ..." respectively.
- ▶ $\prec_{\mathcal{L}_{stat}^{\aleph_0, II}}$ and $\prec_{\mathcal{L}_{stat}^{\aleph_0, II}}^-$ are defined as before.
- ▶ Also, we regard the logic $\mathcal{L}_{stat}^{\aleph_0, II}$ as equipped with $\prec_{\mathcal{L}_{stat}^{\aleph_0, II}}$.
- ▶ The logic $\mathcal{L}_{stat}^{\aleph_0}$ equipped with $\prec_{\mathcal{L}_{stat}^{\aleph_0, II}}^-$ is denoted with $\mathcal{L}_{stat}^{\aleph_0, II-}$.

- ▶ \mathcal{L}^{II} denotes the (monadic, full) second-order logic with second-order variables X, Y, Z etc. running over all subsets of the underlying set of a structure. In addition to the constructs of the first-order logic, we have the symbol ε as a logical binary predicate and allow the expression " $x \varepsilon X$ " for a first order variable x and a second-order variable X as an atomic formula. We also allow the quantification of the form " $\exists X$ " (and its dual " $\forall X$ ") over the second-order variables X .
- ▷ The relation symbol ε is interpreted as the (real) element relation and the interpretation of the quantifier $\exists X$ in \mathcal{L}^{II} is defined by:

$$\mathfrak{A} \models \exists X \varphi(a_0, \dots, a_{m-1}, B_0, \dots, B_{n-1}, X) \quad :\Leftrightarrow$$

there exists a $B \in \mathcal{P}(|\mathfrak{A}|)$ s.t. $\mathfrak{A} \models \varphi(a_0, \dots, a_{m-1}, B_0, \dots, B_{n-1}, B)$

for a first-order structure \mathfrak{A} , an \mathcal{L}^{II} -formula φ in the signature of the structure \mathfrak{A} with $\varphi = \varphi(x_0, \dots, x_{m-1}, X_0, \dots, X_{n-1}, X)$ where x_0, \dots, x_{m-1} and X_0, \dots, X_{n-1}, X are first- and second-order variables, $a_0, \dots, a_{m-1} \in |\mathfrak{A}|$, and $B_0, \dots, B_{n-1} \in \mathcal{P}(|\mathfrak{A}|)$.

$\mathfrak{B} \prec_{\mathcal{L}^{\text{II}}} \mathfrak{A} \iff \mathfrak{B} \models \varphi(b_0, \dots, b_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \dots, b_{n-1})$ holds for all formulas $\varphi = \varphi(x_0, \dots)$ in \mathcal{L}^{II} without free second-order variables, and for all $b_0, \dots, b_{n-1} \in |\mathfrak{B}|$.

- ▷ Exclusion of second-order free variables and parameters in this context is natural because of the following trivial example:

Example 3. Let $\mathfrak{B} \subsetneq \mathfrak{A}$. Let $B = |\mathfrak{B}|$. Then

$$\mathfrak{A} \models \exists x (x \notin B) \quad \text{but} \quad \mathfrak{B} \models \neg \exists x (x \notin B).$$

Theorem 4. (M. Magidor [1971])

$$\text{LSS}(\mathcal{L}^{\text{II}}) = \{\kappa : \kappa \text{ is supercompact, or a limit of supercompact cardinals}\}.$$

- ▶ A cardinal κ is **supercompact** if, for any $\lambda \geq \kappa$, there are transitive class M and elementary embedding $j : V \rightarrow M$ s.t. κ is the smallest ordinal moved by j (**critical point of j** : we denote these conditions as $j : V \overset{\prec}{\rightarrow}_{\kappa} M$), $j(\kappa) > \lambda$ and $[M]^\lambda \subseteq M$.

[Back to the proof of Proposition 12.](#)

A slight modification of Magidor's theorem

Theorem 4. (M. Magidor [1971])

$$\text{LSS}(\mathcal{L}^{\text{II}}) = \{\kappa : \kappa \text{ is supercompact, or a limit of supercompact cardinals}\}.$$

- ▷ Let \mathcal{L}^{HO} denote the higher order logic that is the union of n th order logics for all $n \in \omega$.
- ▷ Note that, if \mathcal{L}' has more expressive power than \mathcal{L} then we have $\text{LSS}(\mathcal{L}') \subseteq \text{LSS}(\mathcal{L})$.

Corollary 5. (M. Magidor [1971])

$$\begin{aligned} \text{LSS}(\mathcal{L}^{\text{II}}) &= \text{LSS}(\mathcal{L}^{\text{HO}}) \\ &= \{\kappa : \kappa \text{ is supercompact, or a limit of supercompact cardinals}\}. \end{aligned}$$

$$\text{Proof. } \text{LSS}(\mathcal{L}^{\text{II}}) \underbrace{\supseteq}_{\text{by the remark above}} \text{LSS}(\mathcal{L}^{\text{HO}}) \underbrace{\supseteq}_{\text{by a modification of the proof of Theorem 4. "}\supseteq\text{"}} \{\kappa : \dots\}$$

$$\underbrace{\supseteq}_{\text{Theorem 4. "}\subseteq\text{"}} \text{LSS}(\mathcal{L}^{\text{II}})$$

Theorem 4. " \subseteq "

□ (Corollary 8)

The spectrum of compactness numbers of a logic

► For a logic \mathcal{L} , the compactness spectrum of \mathcal{L} is defined as:

$CS(\mathcal{L}) := \{\kappa \in \text{Card} : \text{for any } \mathcal{L}\text{-theory } T \text{ (possibly of an uncountable signature), of size } \kappa, T \text{ is satisfiable if and only if all } S \in [T]^{<\kappa} \text{ are satisfiable}\}.$

► The strong compactness number of a logic \mathcal{L} is defined as:

$scn(\mathcal{L}) := \min(\{\kappa \in \text{Card} : \text{for any } \mathcal{L}\text{-theory } T \text{ (possibly of an uncountable signature) of any size, } T \text{ is satisfiable if and only if all } S \in [T]^{<\kappa} \text{ are satisfiable}\}).$

Lemma 6. For any logic \mathcal{L} ,

- (1) $\{\kappa \in \text{Card} : scn(\mathcal{L}) \leq \kappa\} \subseteq CS(\mathcal{L}).$
- (2) $\min(CS(\mathcal{L})) \leq scn(\mathcal{L}).$

Theorem 7. (M. Magidor [1971], see also Theorem 23.4 in [Kanamori])

For a cardinal $\kappa > \aleph_0$, the following are equivalent:

- (a) κ is extendible.
- (b) $\kappa = \text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$.

- ▶ A cardinal κ is **extendible** if, for any $\eta > 0$, there is $j : V_{\kappa+\eta} \xrightarrow{\sim}_{\kappa} V_{\zeta}$ for some ζ with $\eta < j(\kappa)$.
- ▷ In the definition above, the condition “ $\eta < j(\kappa)$ ” can be dropped: (Proposition 23.15 in [Kanamori]). Also, if $\eta > \kappa$, we may replace $\kappa + \eta$ by η . If the condition above holds for an η we say that κ is **η -extendible**.
- ▶ For a cardinal κ , $\mathcal{L}_{\kappa, \omega}^{\text{II}}$ is the logic defined as \mathcal{L}^{II} but also allowing conjunction and disjunction of $< \kappa$ many formulas (with the restriction that the number of free variables in such conjunction or disjunction is always kept finite).

Proof of Theorem 7: “(a) \Rightarrow (b)”: Suppose that κ is extendible.

- ▶ We first show $\kappa \geq \text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$.
- ▷ Suppose that T is a set of sentences in $\mathcal{L}_{\kappa, \omega}^{\text{II}}$ of size $\lambda \geq \kappa$ s.t. T is $< \kappa$ -satisfiable. We want to show that T is satisfiable.
- ▷ We may assume that all the symbols used in T are elements of λ and the coding of formulas is done in an appropriate way.
- ▶ Let $j : V_\lambda \xrightarrow{\sim}_{\kappa} V_\zeta$ for some ζ and $\lambda < j(\kappa)$. By Theorem 4 and since κ is supercompact (Proposition 23.6 in [Kanamori]), all $T_0 \in [T]^{< \kappa}$ has a model of size $< \kappa$. In particular $V_\lambda \models$ “ T is $< \kappa$ -satisfiable”.
- ▶ $V_\zeta \models$ “ $j''T \in [j(T)]^{< j(\kappa)}$ ”. By elementarity of j ,
 $V_\zeta \models$ “ $j''T$ has a model (of size $< j(\kappa)$)”.

Let $\mathfrak{A} \in V_\zeta$ be a structure s.t. $V_\zeta \models$ “ $\mathfrak{A} \models j''T$ and $\|\mathfrak{A}\| < j(\kappa)$ ”. Then $\mathfrak{A} \models j''T$ (V_ζ interprets model relation of $\mathcal{L}_{\kappa, \omega}^{\text{II}}$ correctly at the structure \mathfrak{A} because of $\mathcal{P}(\lambda)^{V_\zeta} = \mathcal{P}(\lambda)^V$ etc.). By renaming the interpretations of the non-logical symbols $\subseteq j''\lambda$ by corresponding symbols $\subseteq \lambda$ we obtain a model \mathfrak{A}^* of T from \mathfrak{A} .

- ▶ Thus, T is satisfiable.

- ▶ To show that $\kappa \leq \text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$, note that $\text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$ exists by the proof above.
- ▷ So it is enough to show that any $\mu < \kappa$ is not $\text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$.
Suppose $\mu < \kappa$ and consider the theory

$$T := \{\forall x (\bigvee_{\alpha < \mu} x \equiv c_\alpha)\} \cup \{d \not\equiv c_\alpha : \alpha < \mu\}.$$

Then T is not satisfiable but any subset of T of size $< \mu$ is satisfiable. In particular, μ cannot be $\text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$.

“(b) \Rightarrow (a)”: Suppose that $\kappa = \text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$.

- ▶ Let φ^* be the \mathcal{L}^{II} -sentence in the language $\{\in\}$ of set theory s.t.

$$\langle |\mathfrak{A}|, \in^{\mathfrak{A}} \rangle \models \varphi^* \Leftrightarrow \in^{\mathfrak{A}} \text{ is well-founded and extensional binary relation and } mc(\langle |\mathfrak{A}|, \in^{\mathfrak{A}} \rangle) = \langle V_\gamma, \in \rangle \text{ for some } \gamma$$

where mc denotes the Mostowski collapse function.

For regular $\lambda > \kappa$, let

$$\begin{aligned}
 T := & \{\varphi^*\} \cup \{\varphi(j_{\sqcup}(a_0), \dots) : \varphi \text{ is a first-order formula in the} \\
 & \text{language of set theory,} \\
 & a_0, \dots \in V_\lambda \text{ and } V_\lambda \models \varphi(a_0, \dots)\} \\
 & \cup \{\text{"}\alpha \text{ is an ordinal"} : \alpha < \kappa + 1\} \\
 & \cup \{\forall x (x \sqsubseteq \alpha \leftrightarrow \bigvee_{\beta < \alpha} x \equiv \beta) : \alpha < \kappa\} \\
 & \cup \{j_{\sqcup}(\kappa) > \kappa\}.
 \end{aligned}$$

- ▶ It is easy to see that T is $< \kappa$ -satisfiable.

- ▷ By the choice of κ , it follows that T is satisfiable. A model of T witnesses that κ is λ -extendible. Since this holds for arbitrary λ , it follows that κ is extendible. \square (Theorem 7)

Theorem 8. (M. Magidor [1971]) Suppose that κ is the least extendible cardinal. Then we have $\text{scn}(\mathcal{L}^{\text{II}}) = \text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}}) = \kappa$.

Proof. (1) $\text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}}) = \kappa$ follows from Theorem 7.

- ▶ (2) $\text{scn}(\mathcal{L}^{\text{II}}) \leq \text{scn}(\mathcal{L}_{\kappa, \omega}^{\text{II}})$ since $\mathcal{L}_{\kappa, \omega}^{\text{II}}$ extends \mathcal{L}^{II} .
- ▷ Thus it is enough to show that $\kappa^* := \text{scn}(\mathcal{L}^{\text{II}})$ is extendible.
- ▶ By (1) and (2), there is no extendible cardinal below κ^* . Thus, for each $\mu < \kappa^*$, we can choose a cardinal λ_μ s.t. μ is not λ_μ -extendible. Note that it follows that μ is not ξ -extendible for any $\xi \geq \lambda_\mu$. Let $\lambda > \kappa^*$ be s.t.

$$\lambda \geq \sup(\{\lambda_\mu : \mu < \kappa^*\}).$$

► Let

$$\begin{aligned}
 T := & \{\varphi^*\} \cup \{\varphi(j_{\sqcup}(a_0), \dots) : \varphi \text{ is a first-order formula in the} \\
 & \text{language of set theory,} \\
 & a_0, \dots \in V_\lambda \text{ and } V_\lambda \models \varphi(a_0, \dots)\} \\
 & \cup \{\text{"}\alpha \text{ is an ordinal"} : \alpha < \kappa^* + 1\} \\
 & \cup \{\alpha < \beta : \alpha < \beta < \kappa^* + 1\} \\
 & \cup \{j(\kappa^*) > \kappa^*\}.
 \end{aligned}$$

► Again T is $< \kappa^*$ -satisfiable. By the definition of κ^* , it follows that T has a model.

▷ A model \mathfrak{A} of T witnesses the λ -extendibility of some $\mu \leq \kappa^*$. By the choice of λ however it is only possible that κ^* is λ -extendible.

□ (Theorem 8)

Examples showing non-trivial situations

- ▶ Suppose that κ is extendible and λ ($\kappa < \lambda$) is the first supercompact cardinal above κ .
This is stronger than just one extendible cardinal.
See [Kanamori] Exercise 23.9.

- ▷ Let μ ($\lambda < \mu$) be the first inaccessible above λ . We have

$$V_\mu \models \text{ZFC}.$$

- ▷ In V_μ , there is no extendible cardinal since an extendible cardinal would imply cofinally many inaccessible cardinals. By [Theorem 8](#), we have

$$V_\mu \models \text{“scn}(\mathcal{L}^{\text{II}}) \text{ does not exist”}.$$

- ▷ By [Theorem 4](#), sufficiently many witnesses for $\text{CS}(\mathcal{L}^{\text{II}}) \cap \lambda$ survive in $V_\lambda \subseteq V_\mu$. In particular

$$V_\mu \models \kappa \in \text{CS}(\mathcal{L}^{\text{II}}).$$

Fun Questions: Is it consistent that $\text{CS}(\mathcal{L}^{\text{II}}) = \emptyset$? (Yes. Cf.: by [Theorem 4](#), it is very easy to obtain a model of $\text{LSS}(\mathcal{L}^{\text{II}}) = \emptyset$)

- ▷ What is (or can be) $\text{CS}(\mathcal{L}^{\text{II}})$ in $V = L$?



Examples showing non-trivial situations (2/2)

- ▶ Let $\kappa_0 < \kappa_1$ be two consecutive extendible cardinals. Let λ , be a supercompact cardinal strictly between κ_0 and κ_1 .
- ▷ Let \mathbb{G} be (V, \mathbb{P}) -generic for a sufficiently distributive p.o. \mathbb{P} of size $< \kappa_1$ forcing a square sequence for a κ strictly between λ and κ_1 .
- ▷ In $V[\mathbb{G}]$, $\text{CS}(\mathcal{L}^{\text{II}}) \cap \lambda$ survives (see [Theorem 4](#)). In particular

$$V[\mathbb{G}] \models \text{“} \min(\text{CS}(\mathcal{L}^{\text{II}})) \leq \kappa_0 \text{”}.$$

- ▷ κ_1 is now the smallest extendible cardinal
Thus, by [Theorem 8](#),

$$V[\mathbb{G}] \models \text{“} \text{scn}(\mathcal{L}^{\text{II}}) = \kappa_1 \text{”}.$$

Note that \square_κ does not hold above a supercompact.

Theorem 11. If κ is σ -closed gen. supercompact, then $\kappa \in \text{LSS}(\mathcal{L}_{stat}^{\aleph_0, \text{II}})$.

Proof.

- ▶ Assume that κ is σ -closed gen. supercompact. Suppose \mathfrak{A} is a structure with $\|\mathfrak{A}\| \geq \kappa$ and $S \in [|\mathfrak{A}|]^{<\kappa}$. W.l.o.g., assume $|\mathfrak{A}| = \|\mathfrak{A}\| = \lambda$.
 - ▷ Let \mathbb{P} be a σ -closed p.o. s.t. for a (V, \mathbb{P}) -generic \mathbb{G} , there are $j, M \subseteq V[\mathbb{G}]$ s.t. $j : V \xrightarrow{\lambda}_{\kappa} M$, $j(\kappa) > \lambda$ and $j''(\lambda^{\aleph_0}) \in M$.
 - ▷ Then $\mathfrak{B} := j(\mathfrak{A}) \upharpoonright j''|\mathfrak{A}| \in M$. Since $j \upharpoonright |\mathfrak{A}| \in M$ we also have $\mathfrak{A} \in M$ and $M \models j \upharpoonright |\mathfrak{A}| : \mathfrak{A} \xrightarrow{\cong} \mathfrak{B}$.
 - ▷ By σ -closedness of \mathbb{P} we have $([|\mathfrak{A}|]^{\aleph_0})^V = ([|\mathfrak{A}|]^{\aleph_0})^M$. Also, all stationary subsets (club subsets resp.) of $([|\mathfrak{A}|]^{\aleph_0})^V$ remain stationary (club resp.) in M .
 - ▶ Thus, $M \models \mathfrak{B} \prec_{\mathcal{L}_{stat}^{\aleph_0, \text{II}}} j(\mathfrak{A})$, $\|\mathfrak{B}\| < j(\kappa)$, $j(S) = j''S \subseteq |\mathfrak{B}|$.
By elementarity, in V , there is $\mathfrak{C} \prec_{\mathcal{L}_{stat}^{\aleph_0, \text{II}}} \mathfrak{A}$ s.t. $\|\mathfrak{C}\| < \kappa$, $S \subseteq |\mathfrak{C}|$.
- Thus $\kappa \in \text{LSS}(\mathcal{L}_{stat}^{\aleph_0, \text{II}})$. □ (Theorem 11.)

Lemma 13. $\mathcal{L}_{stat}^{\aleph_0}$ extends $L(Q)$ (Q_x : there are uncountably many x).
 It follows that $CS(\mathcal{L}_{stat}^{\aleph_0}) \subseteq CS(L(Q))$ and $scn(L(Q)) \leq scn(\mathcal{L}_{stat}^{\aleph_0})$.

Proof. $stat X \exists_x (x \notin X \wedge \dots)$ replaces $Q_x(\dots)$. \square (Lemma 9)

Lemma 14. For all $n \in \omega$ none of $\aleph_n, (\beth_n)^+$ belongs to $CS(L(Q))$.
 In particular, $scn(\mathcal{L}_{stat}^{\aleph_0}) \geq scn(L(Q)) \geq \beth_\omega$.

Proof. We show this for \aleph_2 and $(\beth_1)^+$. The general case can be proved by modifications of the theories below. $\aleph_1 \notin CS(L(Q))$ is trivial.

- The following (non-satisfiable) theory witnesses $\aleph_2 \notin CS(L(Q))$.

$$T_0 := \{ \text{"}E \text{ is a linear ordering"} , \forall x \neg Q_y (y E x) \} \\ \cup \{ c_\alpha \neq c_\beta : \alpha < \beta < \omega_2 \}$$

- The following (non-satisfiable) theory witnesses $(2^{\aleph_0})^+ \notin CS(L(Q))$.

$$T_1 := \{ \neg Q_x N(x), \forall x \forall y (\forall z (N(z) \rightarrow (z \subseteq x \leftrightarrow z \subseteq y)) \rightarrow x \equiv y) \} \\ \cup \{ c_\alpha \neq c_\beta : \alpha < \beta < (2^{\aleph_0})^+ \}$$

- ▶ Lemma 14 shows that a Compactness Spectrum analogue of Theorem 11 is impossible.
- ▶ For a class \mathcal{P} of p.o.s, let

$\text{scn}^{\mathcal{P}}(\mathcal{L}) := \min\{\kappa \in \text{Card} : \text{for any } \mathcal{L}\text{-theory } T, \text{ if } T \text{ is } < \kappa\text{-satisfiable, then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“}\check{T} \text{ is satisfiable in an inner model containing } \check{T}\text{”}\}$

- ▶ For a class \mathcal{P} of p.o.s, a cardinal κ is said to be **generically supercompact by \mathcal{P}** (or **\mathcal{P} -gen. supercompact**, for short) if, for any $\lambda \geq \kappa$, there are $\mathbb{P} \in \mathcal{P}$, (V, \mathbb{P}) -generic \mathbb{G} , j , $M \subseteq V[\mathbb{G}]$ s.t. $V[\mathbb{G}] \models j : V \xrightarrow{\check{\kappa}} M$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Theorem 15. If κ is \mathcal{P} -gen. supercompact, then $\kappa \geq \text{scn}^{\mathcal{P}}(\mathcal{L}^{\text{HO}})$.

Theorem 15. If κ is \mathcal{P} -gen. supercompact, then $\kappa \geq \text{scn}^{\mathcal{P}}(\mathcal{L}^{\text{HO}})$.

Proof. Suppose that T is an \mathcal{L}^{HO} -theory with $|T| = \lambda$ and T is $< \kappa$ -satisfiable. W.l.o.g., we assume that all non-logical symbols appearing in T are elements of λ and the coding of the formulas is done in an appropriate way.

- ▶ Let $\mathbb{P} \in \mathcal{P}$ be s.t., for a (V, \mathbb{P}) -generic \mathbb{G} , there are $j, M \subseteq V[\mathbb{G}]$ with $j : V \xrightarrow{\lambda} M$, $j(\kappa) > \lambda$ and $(*) j''\lambda \in M$.
- ▷ By elementarity, we have $M \models$ “all $T_0 \in [j(T)]^{< j(\kappa)}$ has a model”.
By the closure property $(*)$ of M , $j''T \in M$. Since $M \models$ “ $|j''T| = \lambda < j(\kappa)$ ” and T is obtained from $j''T$ by renaming of non-logical symbols, we have $T \in M$ and $M \models$ “ T has a model”.

□ (Theorem 15)

Thank you for your attention!
ご清聴ありがとうございました。

관심을 가져 주셔서 감사합니다

Gracias por su atención.

Dziękuję za uwagę.

Grazie per l'attenzione.

Dank u voor uw aandacht.

Ich danke Ihnen für Ihre Aufmerksamkeit.

$LSS(\mathcal{L})$ is closed

Lemma 2. For any logic \mathcal{L} , $LSS(\mathcal{L})$ is a closed class of cardinals.

Proof. Suppose that $\langle \kappa_\alpha : \alpha < \delta \rangle$ is a strictly increasing sequence in $LSS(\mathcal{L})$ and $\kappa = \sup_{\alpha < \delta} \kappa_\alpha$. We want to show that $\kappa \in LSS(\mathcal{L})$.

- Suppose that \mathfrak{A} is a structure and $S \subseteq [|\mathfrak{A}|]^{<\kappa}$. Let $\alpha < \delta$ be s.t. $|S| < \kappa_\alpha$. Since $\kappa_\alpha \in LSS(\mathcal{L})$, there is a $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa_\alpha < \kappa$. This shows that $\kappa \in LSS(\mathcal{L})$. \square (Lemma 2)

Back to p.3

On the restriction to countable signatures

Lemma 1. For a logic \mathcal{L} (with natural properties expected to a “logic”), we have

$$\text{LSS}(\mathcal{L}) = \{\mu \in \text{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. “ \subseteq ”: Suppose that $\mu \in \text{LSS}(\mathcal{L})$ and let \mathfrak{A} be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that \mathfrak{A} is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha} \rangle_{n \in \omega, \alpha < \nu}$ where $R_{n,\alpha}$ is an n -ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.

- ▷ Let $R_n := \bigcup_{\alpha < \nu} \{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n \rangle_{n \in \omega}$. Applying our assumption on μ , we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct a submodel \mathfrak{B} of \mathfrak{A} from \mathfrak{B}^- with the same underlying set and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

On the restriction to countable signatures (2/2)

Lemma 1. For a logic \mathcal{L} (with natural properties expected to a “logic”), we have

$$\text{LSS}(\mathcal{L}) = \{\mu \in \text{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. “ \subseteq ”: Suppose that $\mu \in \text{LSS}(\mathcal{L})$ and let \mathfrak{A} be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that \mathfrak{A} is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha} \rangle_{n \in \omega, \alpha < \nu}$ where $R_{n,\alpha}$ is an n -ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.

Let $R_n := \bigcup_{\alpha < \nu} \{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n \rangle_{n \in \omega}$. Applying our assumption on μ , we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct an \mathcal{L} -elementary submodel \mathfrak{B} of \mathfrak{A} from \mathfrak{B}^- with the same underlying set.

“ \supseteq ”: Suppose now that μ is in the set on the right side of the equality. Let \mathfrak{A} be a structure of size $\geq \mu$ with a countable signature, and $S \in [|\mathfrak{A}|]^{<\mu}$.

Let $\mathfrak{A}^+ = \langle \mathfrak{A}, a \rangle_{a \in S}$. Applying the assumption on μ , we obtain $\mathfrak{B}^+ \prec_{\mathcal{L}} \mathfrak{A}^+$ of size $< \mu$. Denoting by \mathfrak{B} the \mathfrak{B}^+ reduced to the original language, we have $\|\mathfrak{B}\| < \mu$, $S \subseteq |\mathfrak{B}|$ and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

□ (Lemma 1)