On algebraic and geometrical characterizations of linear mappings

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(2020 年 07 月 29 日 (00:39 JST) version)

2020 年 7 月 7 日 (JST, 於 Kobe Set Theory Seminar)

This presentation is typeset by upLaTeX with beamer class.
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The research is partially supported by
Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717

Additive functions on $\mathbb{R}$

- A function $f : \mathbb{R} \to \mathbb{R}$ is said to be additive if

  $f(a+b) = f(a) + f(b)$ holds for all $a, b \in \mathbb{R}$.

- If $f : \mathbb{R} \to \mathbb{R}$ is additive, then f is $\mathbb{Q}$-linear.
- If $f : \mathbb{R} \to \mathbb{R}$ is additive and f is discontinuous at one point, then f is everywhere discontinuous.
- Any monoton $f : \mathbb{R} \to \mathbb{R}$ is discontinuous at most on countably many points.

Proposition 1. For any additive $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent:

(a) f is an $\mathbb{R}$-linear mapping; (b) f is continuous;
(c) f is monotone.

Additive functions on $\mathbb{R}^n$

- For $\mathbb{R}$-linear spaces X, Y, a mapping $f : X \to Y$ is said to be **additive** if
  $f(\mathbb{a} + \mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$ holds for all $\mathbb{a}, \mathbb{b} \in X$.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$ is additive, then f is $\mathbb{Q}$-linear.

Proposition 2. For any additive $f : \mathbb{R}^n \to \mathbb{R}^m$, the following are equivalent:

(a) f is an $\mathbb{R}$-linear mapping; (b) f is continuous;

- The condition "(b) f is continuous" in propositions 1, 2 can be still improved as follows:

Theorem 3. For any additive $f : \mathbb{R}^n \to \mathbb{R}^m$, the following are equivalent:

(a) f is an $\mathbb{R}$-linear mapping; (b) f is continuous;
(c) (Sierpiński) f is a measurable function;
(d) (Steinhaus) f is a Baire function.

ZFC **vers.** ZF + DC + ?

Proposition 4. (ZFC, Hamel) There is a discontinuous additive function $f : \mathbb{R} \to \mathbb{R}$.

- Proposition 3. implies the following:

Corollary 5. (ZF + AD) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping if and only if it is additive.

Corollary 6. ZF + DC + "any function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear mapping if and only if it is additive" is equiconsistent with ZF.

Proof. By Propositon 3. and Shelah's "Can you take Solovay's inaccessible away?" □

Additive and multiplicative functions on $\mathbb{R}$

- A function $f : \mathbb{R} \to \mathbb{R}$ is said to be multiplicative if

  $f(ab) = f(a)f(b)$ holds for all $a, b \in \mathbb{R}$.

Lemma 7. (Kuczma [1], Theorem 14.4.1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is additive and multiplicative. Then f is either the constant zero function or the identity function.

証明.

Theorem 8. For $\mathbb{R}$-linear spaces X, Y and an additive mapping $f : X \to Y$, if there is a function $\varphi : \mathbb{R} \to \mathbb{R}$ s.t.

$$f(r\mathbb{a}) = \varphi(r)f(\mathbb{a}) \text{ for all } \mathbb{a} \in X \text{ and } r \in \mathbb{R},$$

then f is an $\mathbb{R}$-linear mapping.

証明.

[1] Marek Kuczma, An introduction to the theory of functional equations and inequalities, Second Edition (ed. by Attila Gilányi), Birkhäuser (2009).

Thm.13

Points and lines

- For an $\mathbb{R}$-linear space X, $L \subseteq X$ is a line if and only if it is a 1-dimensional affine subspace of X. Thus L is a line if there are $\mathbb{a}$, $\mathbb{b} \in X$ s.t. $L = \mathbb{R}\mathbb{a} + \mathbb{b} = \{r\mathbb{a} + \mathbb{b} : r \in \mathbb{R}\}$.
- $P \subseteq X$ is a point if it is a 0-dimensional affine subspace of X. That is, if P is a singleton.

Lemma 9. Suppose that X, Y are $\mathbb{R}$-linear spaces, and $f : X \to Y$ is an $\mathbb{R}$-linear mapping. Then, for each line $L \subseteq \mathbb{R}$, $f''L$ is either a line or a point in Y, and if $f''L$ is a line, then $f \restriction L$ is 1-1.

Lemma 10. For $\mathbb{R}$-linear spaces X, Y, there is a non-linear mapping $f : X \to Y$ s.t. $f(\mathbb{0}_X) = \mathbb{0}_Y$, $L \subseteq X$ either to a line or to a point for each line $L \subseteq X$, and $f \restriction L$ is 1-1 for a line $L \subseteq X$ if $f''L$ is a line.

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An Open Question and a partial Answer

Problem 11. Suppose that X, Y are $\mathbb{R}$-linear spaces and $f : X \to Y$. If (1) $f(\mathbb{0}_X) = \mathbb{0}_Y$, (2) for any line $L \subseteq X$, $f''L$ is either a point or a line in Y, (3) there are $\mathbb{a}, \mathbb{b} \in X$ s.t. $f(\mathbb{a})$ and $f(\mathbb{b})$ are independent in Y, does it follow that f is a $\mathbb{R}$-linear mappings?

- The following theorem is often cited as the Fundamental Theorem of Affine Geometry.

  - For $\mathbb{R}$-linear spaces X, Y, $g : X \to Y$ is an $\mathbb{R}$-affine mapping if there are $\mathbb{R}$-linear mapping $f : X \to Y$ and $\mathbb{b} \in Y$ s.t. g is defined by $g(\mathbb{a}) = f(\mathbb{a}) + \mathbb{b}$.

Theorem 12. (Fundamental Theorem of Affine Geometry) For any $n \in \omega \setminus 2$, if a bijection $g : \mathbb{R}^n \to \mathbb{R}^n$ send any line to a line, then g is an $\mathbb{R}$-affine mapping. In particular, such g is a linear mapping if $g(\mathbb{0}_X) = \mathbb{0}_Y$. □

- We show that a statement, which is between the statements in Problem 11 and Theorem 12, characterizes affineness of $f : X \to Y$ with $f''X$ not being included in a line in Y.

A positive partial answer

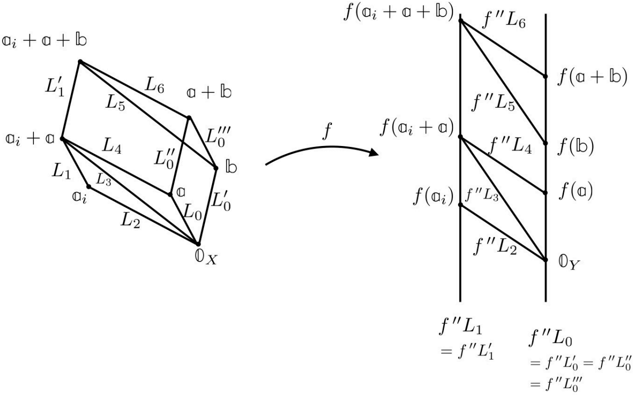
Theorem 13. Suppose that X, Y are $\mathbb{R}$-linear spaces and $f : X \to Y$. If (1) $f(\mathbb{0}_X) = \mathbb{0}_Y$, $(2')$ for any line $L \subseteq X$, $f''L$ is either a point or a line in Y, and if $f''L$ is a line then $f \restriction L$ is 1-1; (3) there are $\mathbb{a}_0, \mathbb{a}_1 \in X$ s.t. $f(\mathbb{a}_0)$ and $f(\mathbb{a}_1)$ are independent in Y, then f is a $\mathbb{R}$-linear mappings.

A sketch of the proof: We prove that f is additive and there is $\varphi : \mathbb{R} \to \mathbb{R}$ as in Theorem 8.

- To prove the aditivity one of the cases to consider is when $\mathbb{a}$ and $\mathbb{b}$ are linearly independent (in X) but $f(\mathbb{a})$ and $f(\mathbb{b})$ are linearly dependent. In this case there is $i \in 2$ s.t. $f(\mathbb{a}_i)$ is linealy independent form $f(\mathbb{a})$.

A positive partial answer (2/2)

- The constellation of these and some other poins can be put together in the following diagram:



- Some lines on the left side are sent to the same line by f. But enough parallelism servives and this enables to conclude that $f(\mathbb{a} + \mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$. □

Thank you for your attention!

Grazie per l'attenzione!

Proof of Lemma 7.

Lemma 7. (Kuczma [1], Theorem 14.4.1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is additive and multiplicative. Then f is either the constant zero function or the identity function.

Proof. For any $x \in \mathbb{R}$,

(*) if $x \geq 0$, then $f(x) = f((\sqrt{x})^2) = (f(\sqrt{x}))^2 \geq 0$

by multiplicativity of f. By additivity of f, it follows that, for any $x, y \in \mathbb{R}$ with $x \leq y$,

$$f(y) = f(x + (y - x)) = f(x) + f(y - x) \geq f(x).$$

Thus f is a monotone function.

- Since f is additive, by Proposition 1, there is $c \in \mathbb{R}$ s.t. $f(x) = cx$ holds for all $x \in \mathbb{R}$. We have

$$c = f(1) = f(1 \cdot 1) = f(1)f(1) = c^2.$$

- Since $f(1) \geq 0$ by (*), it follows that $c = 0$ or $c = 1$. If $c = 0$, f is the constant function $f(x) = 0$ for all $x \in \mathbb{R}$. If $c = 1$, $f = id_{\mathbb{R}}$. □

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Proof of Theorem 8.

Theorem 8. For $\mathbb{R}$-linear spaces X, Y and an additive mapping $f : X \to Y$, if there is a function $\varphi : \mathbb{R} \to \mathbb{R}$ s.t.

$$f(r\mathbb{a}) = \varphi(r)f(\mathbb{a}) \text{ for all } \mathbb{a} \in X \text{ and } r \in \mathbb{R},$$

then f is an $\mathbb{R}$-linear mapping.

Proof. If $f''X = \{\mathbb{0}_Y\}$, then f is a linear mapping. Thus, we may assume that $f''X \neq \{\mathbb{0}_Y\}$. Then we have $\varphi(1) = 1$. Hence, by Lemma 7, it is enough to show that φ is additive and multiplicative.

- To show that φ is additive, suppose that $r, s \in \mathbb{R}$. Let $\mathbb{a} \in X$ be s.t. $f(\mathbb{a}) \neq \mathbb{0}_Y$. By additivity of f, we have $\varphi(r+s)f(\mathbb{a}) = f((r+s)\mathbb{a}) = f(r\mathbb{a}+s\mathbb{a}) = f(r\mathbb{a})+f(s\mathbb{a}) = \varphi(r)f(\mathbb{a})+\varphi(s)f(\mathbb{a}) = (\varphi(r)+\varphi(s))f(\mathbb{a})$. It follows that $\varphi(r+s) = \varphi(r)+\varphi(s)$.
- Multiplicativity of φ can be shown similarly: Suppose $r, s \in \mathbb{R}$, and let $\mathbb{a} \in X$ be s.t. $f(\mathbb{a}) \neq \mathbb{0}_Y$. Then we have $\varphi(rs)f(\mathbb{a}) = f(rs\mathbb{a}) = \varphi(r)f(s\mathbb{a}) = \varphi(r)\varphi(s)f(\mathbb{a})$. It follows that $\varphi(rs) = \varphi(r)\varphi(s)$. □

Proof of Lemma 10.

Lemma 10. For any $\mathbb{R}$-linear spaces X, Y, there is a non linear mapping $f : X \to Y$ s.t. $f(\mathbb{0}_X) = \mathbb{0}_Y$, $L \subseteq X$ either to a line or to a point for each line $L \subseteq X$, and $f \restriction L$ is 1-1 for a line $L \subseteq X$ if $f''L$ is a line.

Proof. Suppose that $\mathbb{e}_0 \in X \setminus \{\mathbb{0}_X\}$ and $\mathbb{d}_0 \in Y \setminus \{\mathbb{0}_Y\}$. Let B be a linear basis of X with $\mathbb{e}_0 \in B$. For $\mathbb{x} \in X$, let $\varphi(\mathbb{x}) \in \mathbb{R}$ be the $\mathbb{e}_0$-coordinate of $\mathbb{x}$ w.r.t. B. That is, let $\varphi(\mathbb{x}) \in \mathbb{R}$ be s.t. there is a linear combination $\mathbb{c}$ of elements of $B \setminus \{\mathbb{e}_0\}$ s.t. $\mathbb{x} = \varphi(\mathbb{x})\mathbb{e}_0 + \mathbb{c}$.

- Let $\psi : \mathbb{R} \to \mathbb{R}$ be any non-linear bijective function with $\psi(0) = 0$.
- Let $f : X \to Y$ be defined by $f(\mathbb{x}) = \psi(\varphi(\mathbb{x}))\mathbb{d}_0$. Clearly f is not a linear mapping. $f(\mathbb{0}_X) = \mathbb{0}_Y$ by the definition of f.
- For a line $L \subseteq X$ with $L = \mathbb{R}\mathbb{b} + \mathbb{a}$ for some $\mathbb{a}$, $\mathbb{b} \in X$, if $\varphi(\mathbb{b}) = 0$ then $f''L = \{\psi(\varphi(\mathbb{a}))\mathbb{d}_0\}$. Otherwise, by bijectivity of ψ, we have $f''L = \mathbb{R}\mathbb{d}_0$, and $f \restriction L$ is 1-1. □