A/the (possible) solution of the Continuum Problem and the existence of Laver generic large cardinal

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The solution of the Continuum Problem

A ~~The~~ solution of the Continuum Problem

- The continuum is either $\aleph_1$ or $\aleph_2$ or very large.

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  ▷ Provided that a reasonable, and sufficiently strong reflection principle should hold.

- The continuum is either $\aleph_1$ or $\aleph_2$ or very large.

  ▷ Provided that a Laver-generically supercompact cardinal should exist. Under a Laver-generically supercompact cardinal, in each of the three scenarios, the respective reflection principle in the sense of above also holds.

The results discussed in the following slides ...

are to be found in the joint papers with André Ottenbereit Maschio Rodriques and Hiroshi Sakai:

[1] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, Archive for Mathematical Logic (2020). https://fuchino.ddo.jp/papers/SDLS-x.pdf

[2] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, to appear.
https://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[3] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, III — mixed support iteration, submitted.
https://fuchino.ddo.jp/papers/SDLS-III-x.pdf

[4] Sakaé Fuchino, and André Ottenbereit Maschio Rodriques, Reflection principles, generic large cardinals, and the Continuum Problem, to appear in the Proceedings of the Symposium on Advances in Mathematical Logic 2018.
https://fuchino.ddo.jp/papers/refl_principles_gen_large_cardinals_continuum_problem-x.pdf

The size of the continuum

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  - provided that a "reasonable", and sufficiently strong reflection principle should hold.

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Theorem A. $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* CH. 証明
Actually $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ is equivalent with Sean Cox's Diagonal Reflection Principle for internal clubness + CH.

Theorem B. (a) $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0} = \aleph_2$. 証明
(b) $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to Diagonal Reflection Principle for internal clubness* **(c)** $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *is equivalent to* $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2) + \neg\mathrm{CH}$. 証明

Theorem C. $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0}$ *is very large (e.g. weakly Mahlo, weakly hyper Mahlo, etc.)* 証明

Theorem 1. (B. König; S.F., Ottenbreit, and Sakai [1])
The following are equivalent:

(a) κ^+ is generically supercompact for $<\kappa$-closed p.o.s.

(b) $2^{<\kappa} = \kappa$ and $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ holds.

Lemma 2. If κ^* is supercompact and $\kappa < \kappa^*$ is regular, then, for $\mathbb{P} = \mathrm{Col}(\kappa, \kappa^*)$ and, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have:

$\mathsf{V}[\mathbb{G}] \models \kappa^* = \kappa^+$, $\kappa^{<\kappa} = \kappa$, and κ^+ is a generically supercompact cardinal for $<\kappa$-closed p.o.s.

▷ Note that, for $\kappa < \kappa'$, if δ is generically supercompact for κ'-closed p.o.s, then δ is generically supercompact for κ-closed p.o.s.

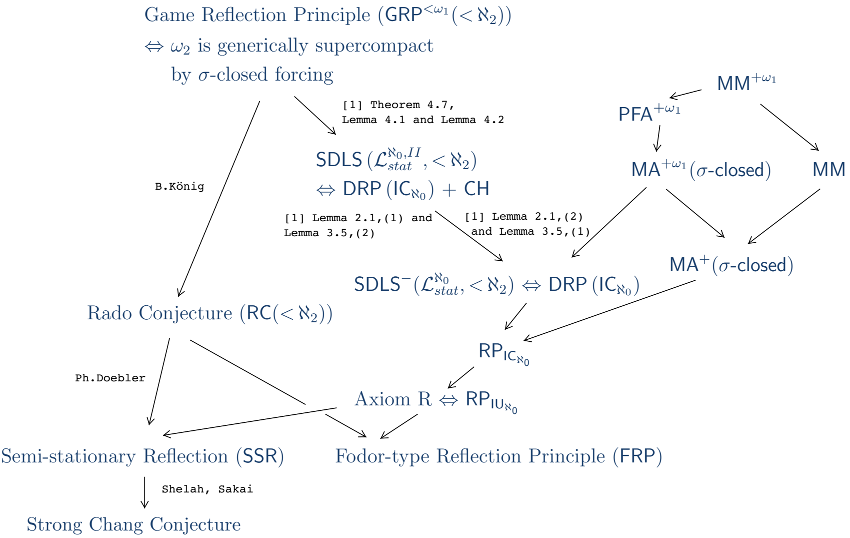
Theorem 1. (B. Köng; S.F., Ottenbreit, and Sakai [1])
(1) Suppose that $\kappa > \aleph_1$ is a regular cardinal s.t. $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$. Then $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$.
(2) For a regular uncountable cardinal $\kappa > \aleph_1$, $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies the Rado's Conjecture $\mathsf{RC}(<\kappa)$ with reflection point $<\kappa$.

Theorem 2. (B. König) $\mathsf{GRP}^{<\omega_1}(<\omega_2)$ implies CH.

- More generally for regular $\kappa > \aleph_1$, $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies $2^{\aleph_0} < \kappa$ (see [1], Lemma 4.2).

Corollary 3. (1) $\mathsf{GRP}^{<\omega_1}(<\omega_2)$ is equivalent to the statement: ω_2 is a generically supercompact cardinal for σ-closed p.o.s.
(2) $\mathsf{GRP}^{<\omega_1}(<\omega_2)$ implies the Rado Conjecture RC and $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

Generically large cardinals als the untimate reflection principles (3/3)



The size of the continuum ...

- is either $\aleph_1$ or $\aleph_2$ or very large!

- provided that
a reasonable, and sufficiently strong reflection principle
with the reflection point either $\leq \aleph_1$ or $< 2^{\aleph_0}$ should hold.

- The consistency proofs of all of the strong reflection principles in the statement above are obtained by similar arguments.

- By analyzing these proofs, we arrive at the following notion of of Laver-generic supercompactness:

The size of the continuum ...

- is either $\aleph_1$ or $\aleph_2$ or very large!

▷ provided that a strong variant of generic large cardinal should exist.

For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a Laver-generically supercompact for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

(1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

(2) $\mathbb{P}, \mathbb{H} \in M$,

(3) $j''\lambda \in M$.

- κ is Laver-generically superhuge for $\mathcal{P}$ if (3) above is replaced by (3)'' $j''j(\kappa) \in M$.
- κ is Laver-generically super-almost-huge for $\mathcal{P}$ if (3) above is replaced by (3)' $j''\delta \in M$ for all $\delta < j(\kappa)$.

The condition $j''\lambda \in M$ in place of ${}^{\lambda}M \subseteq M$

Lemma. ([2]) *Suppose that* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter for a p.o.* $\mathbb{P} \in \mathsf{V}$ *and* $j : \mathsf{V} \overset{\prec}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$ *s.t., for cardinals* κ, λ *in* V *with* $\kappa \leq \lambda$, $crit(j) = \kappa$ *and* $j''\lambda \in M$.

(1) *For any set* $A \in \mathsf{V}$ *with* $\mathsf{V} \models |A| \leq \lambda$, *we have* $j''A \in M$.

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$.

(3) *For any* $A \in \mathsf{V}$ *with* $A \subseteq \lambda$ *or* $A \subseteq \lambda^2$ *we have* $A \in M$.

(4) $(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, *Thus, if* $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, *then* $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.

(6) $j \restriction A \in M$ *for all* $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.

Consistency of Laver-generically supercompact cardinals

Theorem. ([2]) (1) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for* σ*-closed p.o.s" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically super-almost-huge cardinal for proper p.o.s" is consistent as well.*

証明

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a strongly Laver-generically supercompact cardinal for c.c.c. p.o.s" is consistent as well.*

Proposition. ([2]) (1) *Suppose that* κ *is generically measurable by a* ω_1 *preserving* $\mathbb{P}$*. Then* $\kappa > \omega_1$. 証明

(2) *Suppose that* κ *is Laver-generically supercompact for* ω_1*-preserving* $\mathcal{P}$ *with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$*. Then* $\kappa = \omega_2$. 証明

(3) *Suppose that* $\mathcal{P}$ *is a class of p.o.s containing a p.o.* $\mathbb{P}$ *s.t. any* $(\mathsf{V}, \mathbb{P})$*-generic filter* $\mathbb{G}$ *codes a new real. If* κ *is a Laver-generically supercompact for* $\mathcal{P}$*, then* $\kappa \leq 2^{\aleph_0}$. 証明

(4) *Suppose that* $\mathcal{P}$ *is a class of p.o.s s.t. elements of* $\mathcal{P}$ *do not add any reals. If* κ *is generically supercompact by* $\mathcal{P}$*, then we have* $2^{\aleph_0} < \kappa$. 証明

(5) *Suppose that* κ *is Laver-generically supercompact for* $\mathcal{P}$ *s.t. all* $\mathbb{P} \in \mathcal{P}$ *are ccc and at least one* $\mathbb{P} \in \mathcal{P}$ *adds a real. Then* $\kappa \leq 2^{\aleph_0}$ *holds and* (a) SCH *holds above* $2^{<\kappa}$. (b) *For all regular* $\lambda \geq \kappa$*, there is a* σ*-saturated normal filter over* $\mathcal{P}_\kappa(\lambda)$. (6) *If* κ *is tightly Laver-generically superhuge for ccc, then* $\kappa = 2^{\aleph_0}$.

The trichotomy

Theorem. ([2]) *Suppose that κ is Laver-generically supercompact cardinal for a class $\mathcal{P}$ of p.o.s.*

(A) *If elements of $\mathcal{P}$ are ω_1-preserving and do not add any reals, and $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$, then $\kappa = \aleph_2$ and CH holds. Also, $\mathsf{MA}^{+\aleph_1}(\mathcal{P}, {<}\,\aleph_2)$ holds.*

(B) *If elements of $\mathcal{P}$ are ω_1-preserving and contain all proper p.o.s then $\mathsf{PFA}^{+\omega_1}$ holds and $\kappa = 2^{\aleph_0} = \aleph_2$.*

(C) *If elements of $\mathcal{P}$ are μ-cc for some $\mu < \kappa$ and $\mathbb{P}$ contains a p.o. which adds a reals then κ is fairly large and $\kappa \leq 2^{\aleph_0}$ also $\mathsf{MA}^{+\mu}(\mathcal{P}, {<}\,\kappa)$. holds for any $\mu < \kappa$.*

The trichotomy with SDLS and Laver genericity

Theorem A. *If there exists a Laver-generically supercompact cardinal κ for σ-closed p.o.s, then $\kappa = \aleph_2$ and CH holds. Moreover $MA^{+\aleph_1}(\sigma\text{-closed})$ holds. Thus $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ also holds.*

Theorem B. *If there exists a Laver-generically supercompact cardinal κ for proper p.o.s, then $\kappa = \aleph_2 = 2^{\aleph_0}$. Moreover $\mathsf{PFA}^{+\aleph_1}$ holds. Thus $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ also holds.*

Theorem C. *If there exists a Laver generically supercompact cardinal κ for c.c.c. p.o.s, then $\kappa \leq 2^{\aleph_0}$ and κ is very large (for all regular $\lambda \geq \kappa$, there is a σ-saturated normal ideal over $\mathcal{P}_\kappa(\lambda)$). Moreover $\mathsf{MA}^{+\mu}(ccc, <\kappa)$ for all $\mu < \kappa$ and $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ hold. $\kappa = 2^{\aleph_0}$ is attained if we assume the tightly Laver genrically superhugeness for c.c.c. p.o.s.*

Thank you for your attention.

Grazie per l'attenzione!

Laver generically large cardinals

For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a Laver-generically supercompact for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

(1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

(2) $\mathbb{P}, \mathbb{H} \in M$,

(3) $j''\lambda \in M$.

- κ is Laver-generically superhuge for $\mathcal{P}$ if (3) above is replaced by (3)" $j''j(\kappa) \in M$.
- κ is Laver-generically super-almost-huge for $\mathcal{P}$ if (3) above is replaced by (3)' $j''\delta \in M$ for all $\delta < j(\kappa)$.

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tightly Laver generically superhuge cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a tightly Laver-generically superhuge for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

  (1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

  (2) $\mathbb{P}, \mathbb{H} \in M$,

  (3) $j''j(\kappa) \in M$, and

  (4) $|\,\mathbb{Q}\,| \leq j(\kappa)$.

Proposition にもどる

Proof of Proposition, (4)

Proposition, (4) Suppose that $\mathcal{P}$ is a class of p.o.s s.t. elements of $\mathcal{P}$ do not add any reals. If κ is generically supercompact by $\mathcal{P}$, then we have $2^{\aleph_0} < \kappa$.

Proof. Suppose that $\kappa \leq 2^{\aleph_0}$ and let $\lambda \geq 2^{\aleph_0}$.

- Let $\mathbb{P} \in \mathcal{P}$ be s.t. for some $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ with j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.
- By elementarity, $M \models$"$j(\kappa) \leq (2^{\aleph_0})^M$". Thus

$$(2^{\aleph_0})^V \geq (2^{\aleph_0})^{V[\mathbb{G}]} \geq (2^{\aleph_0})^M \geq j(\kappa) > \lambda \geq (2^{\aleph_0})^{\mathsf{V}}.$$

This is a contradiction.

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Proof of Proposition, (3)

Proposition 3, (3) Suppose that $\mathcal{P}$ is a class of p.o.s containing a p.o. $\mathbb{P}$ which adds a new real. If κ is a Laver-generically supercompact for $\mathcal{P}$, then $\kappa \leq 2^{\aleph_0}$.

Proof. Let $\mathbb{P} \in \mathcal{P}$ be s.t. any generic filter over $\mathbb{P}$ codes a new real. Suppose that $\mu < \kappa$. We show that $2^{\aleph_0} > \mu$. Let $\vec{a} = \langle a_\xi : \xi < \mu \rangle$ be a sequence of subsets of ω. It is enough to show that $\vec{a}$ does not enumerate $\mathcal{P}(\omega)$.

- By Laver-generic supercompactness of κ for $\mathcal{P}$, there are $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, transitive $M \subseteq \mathsf{V}[\mathbb{H}]$ and $j \subseteq M[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\rightarrow} M$, $crit(j) = \kappa$ and $\mathbb{P}, \mathbb{H} \in M$. Since $\mu < \kappa$, $j(\vec{a}) = \vec{a}$.
- Since $\mathbb{H} \in M$ where $\mathbb{G} = \mathbb{H} \cap \mathbb{P}$ and $\mathbb{G}$ codes a new real not in V, we have

  $$M \models \text{“} j(\vec{a}) \text{ does not enumerate } 2^{\aleph_0}\text{”}.$$
- By elementarity, it follows that

  $$\mathsf{V} \models \text{“} \vec{a} \text{ does not enumerate } 2^{\aleph_0}\text{”}.$$

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Proof of Proposition, (2)

Proposition, (2) Suppose that κ is Laver-generically supercompact for ω_1-preserving $\mathcal{P}$ with $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$. Then $\kappa = \omega_2$.

Proof. Suppose that $\kappa \neq \omega_2$. Then, by (1), we have $\kappa > \omega_2$

- Let $\mathbb{Q} \in \mathcal{P}$ be s.t. $\mathbb{P} \lessdot \mathbb{Q}$ for $\mathbb{P} = \mathrm{Col}(\omega_1, \{\omega_2\})$ and s.t., for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$.
- By elementarity, $M \models$"$\underbrace{j((\omega_2)^{\mathsf{V}})}_{=(\omega_2)^{\mathsf{V}}}$ is "ω_2" ". This is a contradiction since $\mathbb{H} \cap \mathbb{P} \in M$ collapes $(\omega_2)^{\mathsf{V}}$ to an ordinal of cardinality $\aleph_1$.

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Proof of Proposition, (1)

Proposition, (1) Suppose that κ is generically measurable by a ω_1 preserving $\mathbb{P}$. Then $\kappa > \omega_1$.

Proof. Suppose that $\kappa \leq \omega_1$. Since $\kappa = \omega$ is impossible, we have $\kappa = \omega_1$.

- Let $\mathbb{P}$ be an ω_1 preserving p.o. and $\mathbb{G}$ a $(\mathrm{V}, \mathbb{P})$-generic filter with $M, j \subseteq V[\mathbb{G}]$ s.t. $j : \mathrm{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$.
- By elementarity we have $M \models$ "$j(\kappa) = \omega_1$".
- Thus $(\omega_1)^V < (\omega_1)^M \leq (\omega_1)^{\mathrm{V}[\mathbb{G}]}$. This is a contradiction to the ω_1 preserving of $\mathbb{P}$.

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Proof of Theorem, (2)

Theorem, (2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically super-almost-huge cardinal for proper p.o.s" is consistent as well.*

Proof. Suppose that κ is a superhuge cardinal. By Corazza [corazza] there is a Laver function $\ell : \kappa \to V_\kappa$ for super almost-hugeness.

- We iterate proper pos κ times with countable support along with the Laver function.
- Let $\mathbb{P}_\kappa$ be the κth stage of the iteration. and let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. To show the Laver-generic super-almost-hugensess, of κ for proper p.o.s, let $\mathbb{Q}$ be a proper p.o.in $\mathsf{V}[\mathbb{G}]$ and $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{Q}$, for $\lambda \geq \kappa$, let $j : \mathsf{V} \overset{\prec}{\to} M$ be s.t. $crit(j) = \kappa$, $j(\kappa) > \lambda$,
(*) ${}^{j(\kappa)>}M \subseteq M$, and (**) $\ell(\kappa) = \underset{\sim}{\mathbb{Q}}$.

Proof of Theorem, (2) (2/2)

- Let $\mathbb{P}^* = j(\mathbb{P}_\kappa)$. Then

  $M \models$" $\mathbb{P}^*$ is the limit of a CS iteration of small proper p.o.s extending the iteration for $\mathbb{P}_\kappa$ with the κ th iterand being $\underset{\sim}{\mathbb{Q}}$"

  by elementarity and by (**). By the closedness (*) of M, the same statement holds in V. Hence $\mathbb{P}^*$ is proper in V and $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{Q}} \lessdot \mathbb{P}^*$.

  Since $\mathbb{P}_\kappa$ is an intermediate stage of proper CS-iteration toward $\mathbb{P}^*$ $\mathbb{R} = \mathbb{P}^*/\mathbb{G}$ is proper in $\mathsf{V}[\mathbb{G}]$ Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}], \mathbb{R})$-generic filter.

- j can be lifted to a super almost huge embedding

  $$\tilde{j} : \mathsf{V}[\mathbb{G}] \to M[\mathbb{G}][\mathbb{H}];\ \underset{\sim}{a}[\mathbb{G}] \mapsto j(\underset{\sim}{a})[\mathbb{G} * \mathbb{H}]$$

  and the lifting witnesses the Laver-generically almost super hugeness of κ in $\mathsf{V}[\mathbb{G}]$.

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Game Reflection Principle

- For any set A and $\mathcal{A} \subseteq {}^{\kappa>}A$, $\mathcal{G}^{\kappa>A}(\mathcal{A})$ is the following game of length κ for players I and II. A match in $\mathcal{G}^{\kappa>A}(\mathcal{A})$ looks like:

$$\begin{array}{c|cccccccccc} \mathrm{I} & a_0 & & a_1 & & a_2 & & \cdots & a_\xi & & \cdots \\ \hline \mathrm{II} & & b_0 & & b_1 & & b_2 & \cdots & & b_\xi & \cdots \end{array} (\xi < \kappa)$$

where a_ξ, $b_\xi \in A$ for $\xi < \kappa$.

- II wins this match if

  $\langle a_\xi, b_\xi \ : \ \xi < \eta \rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi \ : \ \xi < \eta \rangle ^\frown \langle a_\eta \rangle \notin \mathcal{A}$ for some $\eta < \kappa$; or $\langle a_\xi, b_\xi \ : \ \xi < \kappa \rangle \in [\mathcal{A}]$

  where $[\mathcal{A}] = \{f \in: {}^{\kappa}\mathcal{A}, f \restriction \alpha \in \mathcal{A}$ for all $\alpha < \kappa\}$.

- For uncountable regular κ, δ with $\kappa < \delta$, the Game Reflection Princile is defined as:

$\mathrm{GRP}^{<\kappa}(<\delta)$: For any set A of regular cardinality $\geq \delta$, $\mathcal{A} \subseteq {}^{\kappa>}A$, and κ-club $\mathcal{C} \subseteq [A]^{<\delta}$, if the player II has no winning strategy in $\mathcal{G}^{\kappa>A}(\mathcal{A})$, there is $B \in \mathcal{C}$ s.t. the player II has no winning strategy in $\mathcal{G}^{\kappa>B}(\mathcal{A} \cap {}^{\kappa>}B)$.

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Generically supercompact cardinals

For a class $\mathcal{P}$ of p.o.s, a cardinal κ is generically supercompact for $\mathcal{P}$, if for any $\lambda \geq \kappa$ there are $\mathbb{P} \in \mathcal{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. M is transitive, $j : \mathsf{V} \overset{\prec}{\rightarrow} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

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Strong Downward Löwneheim-Skolem Theorem for stationary logic

- $\mathcal{L}^{\aleph_0}_{stat}$ is a weak second order logic with monadic second-order variables X, Y etc. which run over the countable subsets of the underlying set of a structure. The logic has only the weak second order quantifier "*stat*" and its dual "*aa*" (but not the second-order existential (or universal) quantifiers) with the interpretation:

$$\mathfrak{A} \models stat\, X\, \varphi(\ldots, X) \quad :\Leftrightarrow$$
$$\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(\ldots, U)\} \text{ is a stationary subset of } [A]^{\aleph_0}.$$

- For $\mathfrak{B} = \langle B, \ldots \rangle \subseteq \mathfrak{A}$, $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models \varphi(a_0, \ldots, U_0, \ldots) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, \ldots, U_0, \ldots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \ldots, X_0, \ldots)$ and for all $a_0, \ldots \in B$ and for all $U_0, \ldots \in [B]^{\aleph_0}$.

- $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \quad :\Leftrightarrow$
  For any structure $\mathfrak{A} = \langle A, \ldots \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

SDLS($\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2$) **implies** CH.

- Suppose that $\mathfrak{A} = \langle \mathcal{H}(\omega_1), \in \rangle$ and Let $B \in [\mathcal{H}(\omega_1)]^{<\aleph_2}$ be s.t. $\mathfrak{A} \restriction B \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. Then for any $U \in [B]^{\aleph_0}$ we have $\mathfrak{A} \models$ "$\exists x \forall y \, (y \in x \leftrightarrow y \; \varepsilon \; U)$".

- By elementarity we also have $\mathfrak{B} \models$ "$\exists x \forall y \, (y \in x \leftrightarrow y \; \varepsilon \; U)$".

- ▷ It follows that $U \in B$. Thus $[B]^{\aleph_0} \subseteq B$ and $2^{\aleph_0} \leq |\, B \,| \leq \aleph_1$. □

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$\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ implies $2^{\aleph_0} = \aleph_2$.

Proposition 1. $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ *for* $\kappa > \aleph_2$ *implies* $\kappa > 2^{\aleph_0}$.

証明

- Suppose that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds. Then $2^{\aleph_0} \leq \aleph_2$ by the Proposition 1.
- $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_1)$ does not hold since

  "there are uncountably many x s.t. ..."

  is expressible in $\mathcal{L}^{\aleph_0}_{stat}$. [e.g. by $stat\, X\, (\exists x\, (\cdots \wedge x \notin X))$]

  Thus, $2^{\aleph_0} > \aleph_1$. □

Corollary 2. $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ *is inconsistent.*

Proof. Assume $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$. Then $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds. Thus $2^{\aleph_0} = \aleph_2$ by the proof above. But then $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ holds. By Proposition 1. This implies $2^{\aleph_0} = \aleph_1$. This is a contradiction. □

Diagonal Reflection Principle

- (S. Cox) Diagonal Reflection Principle: for a regular cardinal $\theta > \aleph_1$,

  DRP(θ, IC): There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ s.t.

  (1) $M \cap \mathcal{H}(\theta)$ is internally club;

  (2) for all $R \in M$ s.t. R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

- For a regular cardinal $\lambda > \aleph_1$

  $(*)_\lambda$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$, if $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$, is a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$, then there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, for all $a \in M$.

Proposition 1. *TFAE:* (a) *The global version of Diagonal Reflection Principle of S.Cox for internal clubness (i.e.* DRP(θ, IC) *for all regular* $\theta > \aleph_1$*) holds.*

(b) $(*)_\lambda$ *for all regular* $\lambda > \aleph_1$ *holds.*

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A weakening of the Strong Downward Löwenheim-Skolem Theorem

▷ For $\mathfrak{B} = \langle B, \dots \rangle \subseteq \mathfrak{A}$, $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
$\mathfrak{B} \models \varphi(a_0, \dots) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, \dots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \dots)$ without free seond-order variables and for all $a_0, \dots \in B$.

▶ $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \quad :\Leftrightarrow$
For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

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Diagonal Reflection Principle

- (S. Cox) For a regular cardinal $\theta > \aleph_1$:

  DRP(θ, IC): There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ s.t.

  (1) $M \cap \mathcal{H}(\theta)$ is internally club;

  (2) for all $R \in M$ s.t. R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

- For a regular cardinal $\lambda > \aleph_1$

  $(*)_\lambda$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$, if $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$, is a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$, then there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, for all $a \in M$.

Proposition 1. *TFAE:* (a) *The global version of Diagonal Reflection Principle of S.Cox for internal clubness (i.e.* DRP(θ, IC) *for all regular* $\theta > \aleph_1$*) holds.*

(b) $(*)_\lambda$ *for all regular* $\lambda > \aleph_1$ *holds.*

(c) SDLS$^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *holds.*

$$\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0}) \ \Leftrightarrow\ \mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) + \neg\mathrm{CH}.$$

- If $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds then $2^{\aleph_0} = \aleph_2$ by (a). Thus, it follows that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) + \neg\mathrm{CH}$ holds.
- Suppose $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ holds. Then we have $2^{\aleph_0} \leq \aleph_2$ by a theorem of Todorčević already mentioned. Thus, if $2^{\aleph_0} > \aleph_1$ in addition, we have $2^{\aleph_0} = \aleph_2$. Thus $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ follows.

$\square$

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Strong Downward Löwneheim-Skolem Theorem for PKL logic

- $\mathcal{L}^{PKL}_{stat}$ is the weak second-order logic with monadic second order variables X, Y, etc. with built-in unary predicate symbol $\underline{K}$. The monadic seond order variables run over elements of $\mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A)$ for a structure $\mathfrak{A} = \langle A, \underline{K}^{\mathfrak{A}}, ...\rangle$ where we denote

  $\mathcal{P}_S(T) = \mathcal{P}_{|S|}(T) = \{u \subseteq T \;:\; |u| < |S|\}$. The logic has the unique second order quantifier "*stat*" (and its dual).

- The internal interpretation of the quantifier is defined by:

  $\mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, ..., U_0, ..., X) \quad :\Leftrightarrow$
  $\{U \in \mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A) \cap A \;:\; \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\}$ is a stationary subset of $\mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A)$ for $a_0, ... \in A$ and $U_0, ... \in \mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A) \cap A$.

- For $\mathfrak{B} = \langle B, K \cap B, ...\rangle \subseteq \mathfrak{A} = \langle A, K, ...\rangle$, $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models^{int} \varphi(a_0, ..., U_0, ...) \Leftrightarrow \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ...)$ $a_0, ... \in B$ and $U_0, ... \in \mathcal{P}_{K \cap B}(B) \cap B$.

Strong Downward Löwneheim-Skolem Theorem for PKL logic (2/2)

- $\mathrm{SDLS}^{int}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ $:\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, ...\rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$, there is a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

- $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ $:\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, ...\rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$, there are stationarily many substructures $\mathfrak{B}$ of $\mathfrak{A}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

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A Proof of: $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$ implies $2^{\aleph_0}$ is very large.

- For a regular cardinal κ and a cardinal $\lambda \geq \kappa$, $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$ is said to be 2-*stationary* if, for any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there is an $a \in \mathcal{S}$ s.t. $|\kappa \cap a|$ is a regular uncountable cardinal and $\mathcal{T} \cap \mathcal{P}_{\kappa\cap a}(a)$ is stationary in $\mathcal{P}_{\kappa\cap a}(a)$. A regular cardinal κ has the 2-*stationarity property* if $\mathcal{P}_\kappa(\lambda)$ is 2-stationary (as a subset of itself) for all $\lambda \geq \kappa$.

Lemma 1. *For a regular uncountable κ, $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ implies that κ is 2-stationary.*

Lemma 2. *Suppose that κ is a regular uncountable cardinal.*

(1) *If κ is 2-stationary then κ is a limit cardinal.*

(2) *For any $\lambda \geq \kappa$, 2-stationary $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$, and any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there are stationarily many $r \in \mathcal{S}$ s.t. $\mathcal{T} \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.*

(3) *If κ is 2-stationary then κ is a weakly Mahlo cardinal.*

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$\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ for $\kappa > \aleph_2$ implies $\kappa > 2^{\aleph_0}$.

- $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies $2^{\aleph_0} \leq \aleph_2$: it is easy to see that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the reflection principle $\mathrm{RP}(\omega_2)$ in Jech's [millennium-book]. $\mathrm{RP}(\omega_2)$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević).
  - It follows that $\kappa > \aleph_2 \geq 2^{\aleph_0}$.
- Thus, we may assume that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ does not hold. Hence there is a structure $\mathfrak{A}$ s.t., for any $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$, we have $\|\mathfrak{B}\| \geq \aleph_2$. Let $\lambda = \|\mathfrak{A}\|$. W.l.o.g., we may assume that the underlying set of $\mathfrak{A}$ is λ. Let $\mathfrak{A}^* = \langle \mathcal{H}(\lambda^+), \underbrace{\lambda, \ldots,}_{=\mathfrak{A}} \in \rangle$.
- By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$, there is $M \in [\mathcal{H}(\lambda^+)]^{<\kappa}$ s.t. $\mathfrak{A}^* \restriction M \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}^*$. In particular, $\mathfrak{A} \restriction (\lambda \cap M) \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. By the choice of $\mathfrak{A}$, we have $|M| \geq |\lambda \cap M| \geq \aleph_2$.
- By elementarity, there is $C \subseteq [M]^{\aleph_0} \cap M$ which is a club in $[M]^{\aleph_0}$. By a theorem of Baumgartner, it follows that $\kappa > |M| \geq |C| \geq 2^{\aleph_0}$. □

Stationary subsets of $[X]^{\aleph_0}$

- $C \subseteq [X]^{\aleph_0}$ is club in $[X]^{\aleph_0}$ if (1) for every $u \in [X]^{\aleph_0}$, there is $v \in C$ with $u \subseteq v$; and (2) for any countable increasing chain $\mathcal{F}$ in C we have $\bigcup \mathcal{F} \in C$.

- ▷ $S \subseteq [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$ if $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^{\aleph_0}$.

- A set M is internally unbounded if $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$ (w.r.t. $\subseteq$)

- ▷ A set M is internally stationary if $M \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$

- ▷ A set M is internally club if $M \cap [M]^{\aleph_0}$ contains a club in $[M]^{\aleph_0}$.

"Diagonal Reflection Principle" にもどる

Baumgartner's Theorem

- ▷ $\kappa > |\, M \,| \geq |\, \lambda \cap M \,| \geq \aleph_2$
- ▷ there is a club $C \subseteq [M]^{\aleph_0}$ with $C \subseteq M$

Theorem 1 (J.E. Baumgartner). *Let* $\aleph_1 \leq \lambda_0 < \lambda$ *and* λ_0 *be regular. Then any club subset of* $[\lambda]^{<\lambda_0}$ *has cardinality* $\geq \lambda^{\aleph_0}$.

- ▶ $\kappa > |\, M \,| \geq |\, C \,| \geq 2^{\aleph_0}$.

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