Resurrection and Maximality under a/the tightly Laver-generically ultrahuge cardinal

Sakaé Fuchino (渕野 昌)
Kobe University, Japan
https://fuchino.ddo.jp/index.html

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* some additional materials discussed in the talk are put in https://fuchino.ddo.jp/slides/kobe2023-06-05a-pf.pdf

References

[I] S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, Archive for Mathematical Logic, Vol.60, 1-2, (2021), 17–47. https://fuchino.ddo.jp/papers/SDLS-x.pdf

[II] ________________, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Vol.60, 3-4, (2021), 495–523. https://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[Minden] Kaethe Minden, Combining resurrection and maximality, The Journal of Symbolic Logic, Vol. 86, No. 1, (2021), 397–414.

[Tsaprounis 1] Konstantinos Tsaprounis, On resurrection axioms, The Journal of Symbolic Logic, Vol.80, No.2, (2015), 587–608.

[Tsaprounis 2] ________, Ultrahuge cardinals, Mathematical Logic Quarterly, Vol.62, No.1-2, (2016), 1–2.

Outline

Supercompact cardinals

- A cardinal κ is supercompact if, for any $\lambda > \kappa$, there are classes j, M s.t. ① $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, ② $j(\kappa) > \lambda$ and ③ ${}^{\lambda}M \subseteq M$.

- **Notation.** "$j : N \xrightarrow{\prec}_{\kappa} M$" denotes the condition that N and M are transitive (sets or classes); j is a non-trivial elementary embedding of the structure $\langle N, \in \rangle$ into the structure $\langle M, \in \rangle$; $\kappa \in N$, and $crit(j) = \kappa$.

- A supercompact cardinal is a large large cardinal .

- A supercompact cardinal κ enjoys a very strong reflection property down to $< \kappa$: For example:

Proposition 1. Suppose that κ is a supercompact cardinal. For any set X of size $\geq \kappa$ and $\mu < \kappa$, if $\mathcal{S} \subseteq [X]^{\mu}$ is stationary, then there is $Y \subseteq X$ of cardinality $< \kappa$ s.t. $\mathcal{S} \cap [Y]^{\mu}$ is stationary in $[Y]^{\mu}$, and there are stationarily many (actually, normal ultrafilter many) such $Y \in [X]^{<\kappa}$.

Sketch of Proof.

Generically supercompact cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is $\mathcal{P}$-generically supercompact ($\mathcal{P}$-gen. supercompact, for short) if, for every $\lambda > \kappa$, there is $\mathbb{P} \in \mathcal{P}$ s.t., for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. ① $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, ② $j(\kappa) > \lambda$, and ③' $j''\lambda \in M$.

- $\mathcal{P}$-generically supercompact cardinal κ can be a small cardinal. The following constructions of models will be later revisited:

Example 2. Suppose κ is a supercompact cardinal and $\mathcal{P} = \mathrm{Col}(\aleph_1, \kappa)$ (collapsing of all cardinals strictly between $\aleph_1$ and κ by count. conditions). Then for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\kappa = (\aleph_2)^{\mathsf{V}[\mathbb{G}]}$ and $\mathsf{V}[\mathbb{G}] \models$"$\kappa$ is σ-closed-gen. supercompact". ⧉

Example 3. If PFA or MM is forced starting from an almost-huge cardinal κ with an iteration along with an almost-huge Laver-function, then we obtain a model in which κ is the continuum ($= \aleph_2$) and it is proper (or semi-proper)-generic supercompact cardinal. ⧉

Generic supercompactness as a strong reflection principle

Theorem 4. (B. König [B.König]) The following are equivalent:

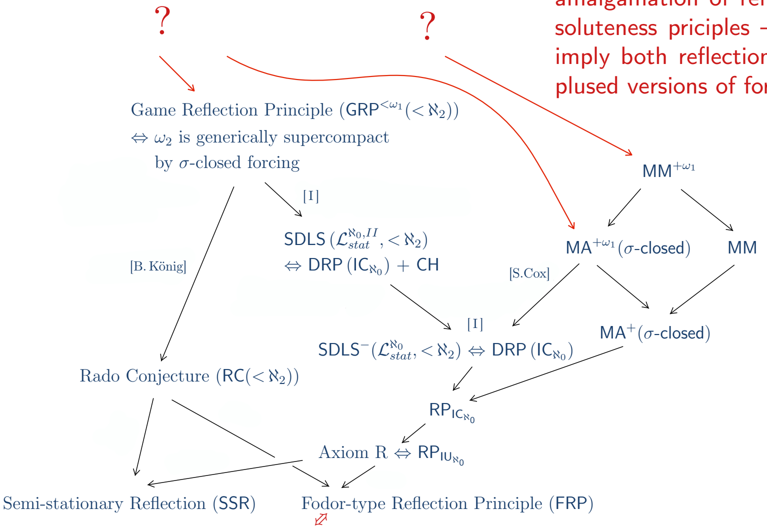
(a) Game Reflection Principle (GRP) holds.

(b) $\aleph_2$ is σ-closed-gen. supercompact.

- GRP is actually a reflection statement about the non-existence of winning strategy of certain games of length ω_1 down to subgames of size $< \aleph_2$.
  - GRP implies (practically) all known reflection principles with reflection down to $< \aleph_2$ available under CH.
- GRP implies Rado's Conjecture (RC) (Bernhard König [B.König]).
- GRP implies strong downward Löwenheim-Skolem Theorem of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ down to $< \aleph_2$ (SDLS($\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}, < \aleph_2$) in the notation of [I]).
- Both RC and SDLS($\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}, < \aleph_2$) imply Fodor-type Reflection Principle (FRP).
- FRP is known to be equivalent to many "mathematical" reflection principles (with reflection down to $< \aleph_2$).

Generic supercompactness as a strong reflection principle (2/2)

- Are there perhaps some yet stronger reflection principle — or some magic amalgamation of reflection and absoluteness priciples — which would imply both reflection and (double-) plused versions of forcing axiom ??



many "mathematical" reflection theorems with reflection down to $<\aleph_2$

Laver-generic large cardinals

- The existence of Laver-generic large cardinal we now introduce is such a reflection and absoluteness principle.

- A (definable) class $\mathcal{P}$ of p.o.s is said to be iterable if ① $\mathcal{P}$ is closed w.r.t. forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$), ② closed w.r.t. restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and, ③ for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

- For an iterable class $\mathcal{P}$ of p.o.s, a cardinal κ is said to be $\mathcal{P}$-Laver-gen. supercompact if, for any $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

  (a) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, (b) $j(\kappa) > \lambda$, and

  (c′) $\mathbb{P} * \underset{\sim}{\mathbb{Q}}, \mathbb{H}, j''\lambda \in M$. (cf. the definition of $\mathcal{P}$-gen. supercompactness)

* The definition of $\mathcal{P}$-Laver-generic supercompactness given here is called strong $\mathcal{P}$-Laver-generic supercompactness in [II].

Laver-generic large cardinals (2/4)

- We can also translate other notions of large cardinal into Laver-generic large cardinal context:

- A cardinal κ is superhuge (super-almost-huge) if, for any $\lambda > \kappa$, there are classes j, M s.t. ① $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$, ② $j(\kappa) > \lambda$ and ③ ${}^{j(\kappa)}M \subseteq M$ $\quad ({}^{j(\kappa)>}M \subseteq M)$.

The upper-half of the "Higher Infinite"

▷ For an iterable class $\mathcal{P}$ of p.o.s, κ is $\mathcal{P}$-Laver-gen. superhuge ($\mathcal{P}$-Laver-gen. super-almost-huge) if, for any $\lambda \geq \kappa$, $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(a) $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$, (b) $j(\kappa) > \lambda$, and

(c′) $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, and $j''j(\kappa) \in M$ ($j''\mu \in M$ for all $\mu < j(\kappa)$).

Laver-generic large cardinals (3/4)

- For an iterable $\mathcal{P}$, a $\mathcal{P}$-Laver-gen. supercompact cardinal ($\mathcal{P}$-Laver-gen. huge cardinal, etc., resp.) is tightly $\mathcal{P}$-Laver-gen. supercompact (tightly $\mathcal{P}$-Laver-gen. huge, etc., resp.) if the condition

  (d) $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of cardinality $\leq j(\kappa)$.

  additionally holds for the elementary embedding j in the definition.

tightly $\mathcal{P}$-Laver-gen. superhuge	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. super-almost-huge	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. supercompact	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. measurable
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\mathcal{P}$-Laver-gen. superhuge	$\Rightarrow$	$\mathcal{P}$-Laver-gen. super-almost-huge	$\Rightarrow$	$\mathcal{P}$-Laver-gen. supercompact	$\Rightarrow$	$\mathcal{P}$-Laver-gen. measurable
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\mathcal{P}$-gen. superhuge	$\Rightarrow$	$\mathcal{P}$-gen. super-almost-huge	$\Rightarrow$	$\mathcal{P}$-gen. supercompact	$\Rightarrow$	$\mathcal{P}$-gen. measurable

- Actually Laver-generic large cardinal is first-order definable (i.e. it has a characterization formalizable in the language of ZFC, [S.F.-Sakai 2]).

  - Thus "Forcing Theorems" are available for arguments with Laver-genericity. Because of this and because an iterable $\mathcal{P}$ is closed under restriction by definition, we may be lazy about the quantification on generic filters like " for a/any $(V, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ ..."

Models of Laver-gen. large cardinal

Theorem 5. (Theorem 5.2, [II]) (1) Suppose κ is supercompact (superhuge, etc., resp.) and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly σ-closed-Laver-gen. supercompact (superhuge, etc., resp.) and CH holds.

(2) Suppose κ is super-almost-huge (superhuge, resp.) with a Laver-function $f : \kappa \to V_\kappa$ for super-almost-hugeness (superhugeness, resp.), and $\mathbb{P}$ is the RCS-iteration for forcing MM along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly semi-proper-Laver-gen. super-almost-huge (superhuge, resp.) and $2^{\aleph_0} = \aleph_2$ holds.

* It seems that the construction does not work with supercompact κ here.

(3) Suppose that κ is supercompact (superhuge, etc. resp.) with a Laver-function $f : \kappa \to V_\kappa$ for supercompactness (superhugeness, etc. resp.), and $\mathbb{P}$ is a FS-iteration for forcing MA along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly ccc-Laver-gen. supercompact (superhuge, etc. resp.). $\kappa = 2^{\aleph_0}$, and κ is very large.

Proof.

Trichotomy Theorem

- Existence of $\mathcal{P}$-Laver-gen. large cardinal for reasonable $\mathcal{P}$ highlights three possible size of the continuum: $\aleph_1$, $\aleph_2$, or very large.

Theorem 6. ([II]) (A) If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of p.o.s such that (a) all $\mathbb{P} \in \mathcal{P}$ are ω_1 preserving, (b) all $\mathbb{P} \in \mathcal{P}$ do not add reals, and (c) there is a $\mathbb{P}_1 \in \mathcal{P}$ which collapses ω_2, then $\kappa = \aleph_2$ and CH holds.

(B) If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of p.o.s such that (a) all $\mathbb{P} \in \mathcal{P}$ are ω_1-preserving, (b′) there is a $\mathbb{P}_0 \in \mathcal{P}$ which add a real, and (c) there is a $\mathbb{P}_1$ which collapses ω_2, then $\kappa = \aleph_2 \leq 2^{\aleph_0}$. If $\mathcal{P}$ contains enough many proper p.o.s then $\kappa = \aleph_2 = 2^{\aleph_0}$ (For the last assertion see the next slide.).

(Γ) If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of p.o.s such that (a′) all $\mathbb{P} \in \mathcal{P}$ preserve cardinals, and (b′) there is a $\mathbb{P}_0 \in \mathcal{P}$ which adds a real, then κ is "very large" and $\kappa \leq 2^{\aleph_0}$. If κ is tightly $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$. □

Forcing Axioms under Laver-genericity

- Suppose that $\mathcal{P}$ is a class of p.o.s, and κ, μ are cardinals.

$\mathsf{MA}^{+\mu}(\mathcal{P}, {<}\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names s.t. $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of ω_1" for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ s.t. $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

$\mathsf{MA}^{++<\mu}(\mathcal{P}, {<}\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names s.t. $|\mathcal{S}| < \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$" for some $\omega < \eta_{\underset{\sim}{S}} \leq \theta_{\underset{\sim}{S}} < \mu$ with $\eta_{\underset{\sim}{S}}$ regular, for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ s.t. $\underset{\sim}{S}[\mathbb{G}]$ is stationary in $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in \mathcal{S}$.

- If $\kappa = \max\{\aleph_2, 2^{\aleph_0}\}$, we drop "${<}\kappa$" and write simply $\mathsf{MA}^{+\mu}(\mathcal{P})$ or $\mathsf{MA}^{++\mu}(\mathcal{P})$.

Forcing Axioms under Laver-genericity

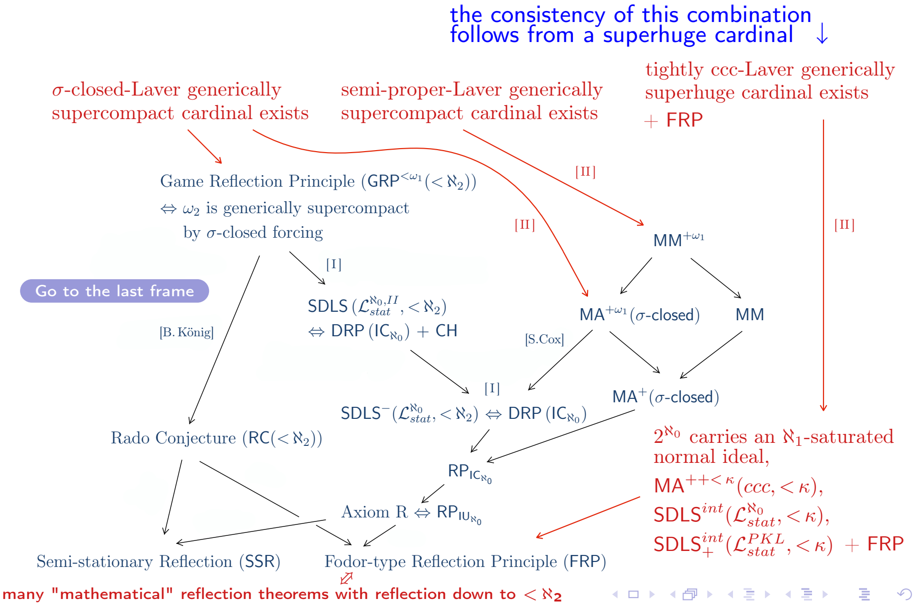
- Clearly $\mathsf{MA}^{++<\omega_2}(\mathcal{P},<\kappa)$ is equivalent to $\mathsf{MA}^{+\omega_1}(\mathcal{P},<\kappa)$.

Theorem 7. (Theorem 5.7 in [II]) (1) For an iterable class $\mathcal{P}$ whose elemetns are all ccc, if $\kappa > \aleph_1$ is $\mathcal{P}$-Laver-generically supercompact, then $\mathsf{MA}^{++<\kappa}(\mathcal{P},<\kappa)$ holds.

(2) If $\aleph_2$ is Laver-generically supercompact for an iterable class $\mathcal{P}$ of p.o.s, then $\mathsf{MA}^{+\omega_1}(\mathcal{P})$ holds.

Proof.

The Trichotomy



Resurrection

- The following Axioms and their variants are introduced and studied by J. Hamkins and T. Johnstone (see [Hamkins-Johnstone 1] , [Hamkins-Johnstone 2]).
- For a class $\mathcal{P}$ of p.o.s and a definition $\mu^\bullet$ of a cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the Resurrection Axiom for $\mathcal{P}$ and $\mathcal{H}(\mu^\bullet)$ is defined by:

$\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, we have $\mathcal{H}(\mu^\bullet)^{\mathsf{V}} \prec \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$.

Resurrection (2/2)

- The following boldface version of the Resurrection Axioms are also considered in [hamkins-johnstone 2] .
- For a class $\mathcal{P}$ of p.o.s and a definition $\mu^\bullet$ of a cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the Resurrection Axiom in Boldface for $\mathcal{P}$ and $\mathcal{H}(\mu^\bullet)$ is defined by:

$\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $A \subseteq \mathcal{H}(\mu^\bullet)$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there is A^* in $\mathsf{V}[\mathbb{H}]$ with $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ and $(\mathcal{H}(\mu^\bullet)^{\mathsf{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$.

- Clearly $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$ implies $\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$.

Theorem 8. For an iterable class of p.o.s $\mathcal{P}$, if $\kappa_{\mathfrak{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$ is tightly $\mathcal{P}$-Laver-gen. superhuge , then $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds.

Proof.

How huge is ultrahuge?

- A cardinal κ is n-huge if there is $j : V \xrightarrow{\prec}_{\kappa} M$ s.t. ${}^{j^n(\kappa)}M \subseteq M$. (Thus, κ is huge $\Leftrightarrow$ κ is 1-huge.)
- A cardinal κ is super n-huge if for any $\lambda > \kappa$ there is $j : V \xrightarrow{\prec}_{\kappa} M$ s.t. $j(\kappa) > \lambda$ and ${}^{j^n(\kappa)}M \subseteq M$.
- A cardinal κ is super n-almost-huge if for any $\lambda > \kappa$ there is $j : V \xrightarrow{\prec}_{\kappa} M$ s.t. $j(\kappa) > \lambda$ and ${}^{j^n(\kappa)>}M \subseteq M$.
- ([Tsaprounis 2]) A cardinal κ is ultrahuge if for any $\lambda > \kappa$ there is $j : V \xrightarrow{\prec}_{\kappa} M$ s.t. $j(\kappa) > \lambda$ and ${}^{j(\kappa)}M, V_{j(\lambda)} \subseteq M$.

Theorem 9. (K. Tsaprounis [Tsaprounis 2] , Theorem 3.4) If κ is 2-almost-huge then there is a normal ultrafilter $\mathcal{U}$ over κ s.t. $\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is ultrahuge”}\} \in \mathcal{U}$.

Laver-gen. ultrahuge cardinal

- We consider the following Laver-gen. variant of ultrahuge cardinal:

▷ For an iterable class $\mathcal{P}$ of p.o.s, a cardinal κ is (tightly) $\mathcal{P}$-Laver gen. ultrahuge, if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{V[\mathbb{H}]} \in M$ (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j(\kappa)$).

Models of Laver-gen. ultrahuge cardinal

- For the construction of models with a Laver-gen. ultrahuge cardinal, we use the following easy lemma:

Lemma 12. Suppose that κ is ultrahuge*. Then there are cofinally many inaccessible cardinals in V.

*super almost-huge is enough see Lemma 2 in the additional slides.

Proof. It is enough to show that the target $j(\kappa)$ of an ultrahuge elementary embedding $j : \mathsf{V} \overset{\prec}{\to}_\kappa M \subseteq \mathsf{V}[\mathbb{H}]$ is inaccessible in $\mathsf{V}[\mathbb{H}]$.

- $M \models$ "$j(\kappa)$ is inaccessible" by elementarity.
- It follows that $(V_{j(\lambda)})^M \models$ "$j(\kappa)$ is inaccessible".
- Since $(V_{j(\lambda)})^M = (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]}$, it follows that $(V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \models$ "$j(\kappa)$ is inaccessible".
- Thus, $\mathsf{V}[\mathbb{H}] \models$ "$j(\kappa)$ is inaccessible". ⧉ (Lemma 12)

Lemma 13. ([Tsaprounis 2], Theorem 5.2) If κ is an ultrahuge cardinal then there is an ultrahuge Laver-function $f : \kappa \to V_\kappa$. ⧉

Models of Laver-gen. ultrahuge cardinal (2/2)

Theorem 14. (1) Suppose κ is ultrahuge and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly σ-closed-Laver-gen. ultrahuge and CH holds.

(2) Suppose κ is ultrahuge with an ultrahuge Laver-function $f : \kappa \to V_\kappa$ and $\mathbb{P}$ is the RCS-iteration for forcing MM along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly semi-proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.

(3) Suppose that κ is ultrahuge with an ultrahuge Laver-function $f : \kappa \to V_\kappa$, and $\mathbb{P}$ is a FS-iteration for forcing MA along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly ccc-Laver-gen. ultrahuge . $\kappa = 2^{\aleph_0}$, and κ is very large .

Proof.

Unbounded resurrection

- The following strengthening of the Resurrection Axiom is introduced in [Tsaprounis 1] :
  - For an iterable class $\mathcal{P}$ of p.o.s, the Unbounded Resurrection Axiom for $\mathcal{P}$ is the following assertion. Remember: $\kappa_{\mathfrak{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$

UR($\mathcal{P}$) : For any $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. $\mathbb{H}$, there are $\lambda^* \in \mathsf{On}$ and $j_0 \in \mathsf{V}[\mathbb{H}]$ s.t. $j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \xrightarrow{\preceq}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, and $j_0(\kappa_{\mathfrak{refl}}) > \lambda$.

- The following tight version of the Unbounded Resurrection Axiom for $\mathcal{P}$ will be also considered.

TUR($\mathcal{P}$) : For any $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. $\mathbb{H}$, there are $\lambda^* \in \mathsf{On}$, and $j_0 \in \mathsf{V}[\mathbb{H}]$ s.t. $j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \xrightarrow{\preceq}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, $j_0(\kappa_{\mathfrak{refl}}) > \lambda$, and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j_0(\kappa_{\mathfrak{refl}})$.

Unbounded resurrection (2/3)

- Both of the principles can be yet extended to boldface versions:

$\mathbb{UR}(\mathcal{P})$: For any $\lambda > \kappa_{\mathfrak{refl}}$, $A \subseteq \mathcal{H}(\lambda)$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. filter $\mathbb{H}$, there are $\lambda^* \in \mathsf{On}$, $A^* \subseteq \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, and $j_0 \in \mathsf{V}[\mathbb{H}]$ s.t. $j_0 : (\mathcal{H}(\lambda)^{\mathsf{V}}, A, \in) \xrightarrow{\preceq}_{\kappa_{\mathfrak{refl}}} (\mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$, and $j_0(\kappa_{\mathfrak{refl}}) > \lambda$.

$\mathbb{TUR}(\mathcal{P})$: For any $\lambda > \kappa_{\mathfrak{refl}}$, $A \subseteq \mathcal{H}(\lambda)$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. filter $\mathbb{H}$, there are $\lambda^* \in \mathsf{On}$, $A^* \subseteq \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, and $j_0 \in \mathsf{V}[\mathbb{H}]$ s.t. $j_0 : (\mathcal{H}(\lambda)^{\mathsf{V}}, A, \in) \xrightarrow{\preceq}_{\kappa_{\mathfrak{refl}}} (\mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$, $j_0(\kappa_{\mathfrak{refl}}) > \lambda$, and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j_0(\kappa_{\mathfrak{refl}})$.

- However, we can prove the equivalence $\mathsf{UR}(\mathcal{P}) \leftrightarrow \mathbb{UR}(\mathcal{P})$ and $\mathsf{TUR}(\mathcal{P}) \leftrightarrow \mathbb{TUR}(\mathcal{P})$.

Unbounded resurrection (3/3)

Theorem 15. For an iterable class $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is (resp. tightly) $\mathcal{P}$-Laver gen. ultrahuge, then $\mathbb{UR}(\mathcal{P})$ (resp. $\mathbb{TUR}(\mathcal{P})$) holds.

Proof. Suppose that $\kappa_{\mathfrak{refl}}$ is (tightly) $\mathcal{P}$-Laver gen. ultrahuge.

- Assume $\lambda > \kappa_{\mathfrak{refl}}$, $A \subseteq \mathcal{H}(\lambda)$, and $\mathbb{P} \in \mathcal{P}$.
- Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. filter $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \xrightarrow{\prec}_{\kappa_{\mathfrak{refl}}} M$, $j(\kappa_{\mathfrak{refl}}) > \lambda$, $\mathbb{P}, \mathbb{H}, V_{j(\lambda)} \in M$
  (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ forcing equivalent to a p.o. of cardinality $j(\kappa_{\mathfrak{refl}})$).

  Note that $\mathcal{H}(j(\lambda))^{\mathsf{V}[\mathbb{H}]} \in M$, and hence $\mathcal{H}(j(\lambda))^M = \mathcal{H}(j(\lambda))^{\mathsf{V}[\mathbb{H}]}$.
- Letting $j_0 := j \restriction \mathcal{H}(\lambda)^{\mathsf{V}}$, $\lambda^* := j(\lambda)$ and $A^* := j(A)$, we have $j_0 : (\mathcal{H}(\lambda), A, \in) \xrightarrow{\prec}_{\kappa_{\mathfrak{refl}}} (\mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$ and $j_0(\kappa_{\mathfrak{refl}}) = j(\kappa_{\mathfrak{refl}}) > \lambda$.
- This shows that $\mathbb{UR}(\mathcal{P})$ ($\mathbb{TUR}(\mathcal{P})$) holds. $\square$ (Theorem 15)

Bounded maximality

- An $\mathcal{L}_{\in}$-formula $\varphi = \varphi(x)$ is a local property of cardinals if, for any limit ordinal δ and a cardinal $\mu < \delta$, we have $(V_\delta \models \varphi(\mu)) \leftrightarrow \varphi(\mu)$ and that this fact is provable in ZFC.

- Being an inaccessible cardinal is a local property of cardinals, as well as being a Mahlo cardinal or being a measurable cardinal. In contrast, being a supercompact cardinal is not a local property of cardinals.

- A local property of cardinals $\varphi = \varphi(x)$ is a local definition of a cardinal if there is provably at most one cardinal which satisfies the formula.

- "The least inaccessible carinal" is a local definition of a cardinal as well as "the least measurable cardinal" but not "the least supercompact cardinal".

Bounded maximality (2/3)

- The property of an ultrahuge cardinal in Lemma 12 can be seen as an "upward reflection". We actually have much stronger upward reflection property of an ultrahuge cardinal:

A Generalization of Lemma 12. If φ is a local notion of cardinal and "κ is ultrahuge" implies $\varphi(\kappa)$ then there are cofinally many cardinals λ with $\varphi(\lambda)$ in V.

- If $\varphi(x)$ is a local definition of a cardinal, we denote the cardinal defined by $\varphi(x)$ with $\kappa^{\bullet}_{\varphi(x)}$, $\mu^{\bullet}_{\varphi(x)}$, etc. or just with $\kappa^{\bullet}$, $\mu^{\bullet}$, etc. if we want to drop the explicit mention of the formula $\varphi(x)$ which defines the term. In the latter notation we identify the term $\kappa^{\bullet}$ with its definition $\varphi(x)$ and say also that $\kappa^{\bullet}$ is a local definition of the cardinal.
- $\beth_{\alpha}(\omega_{\beta})$ for any concretely given finite or countable ordinal α, β is another example of a local definition of a cardinal.

Bounded maximality (3/3)

Theorem 16. Suppose that $\mathcal{P}$ is an iterable class of p.o.s and κ is tightly $\mathcal{P}$-Laver gen. ultrahuge. Then, for any $\mathcal{L}_\in$-formula $\varphi(x_0, \ldots, x_{n-1})$, $a_0, \ldots, a_{n-1} \in \mathcal{H}(\kappa)$, and a local definition $\mu^\bullet$ of a cardinal, if there is $\mathbb{P} \in \mathcal{P}$ s.t.,

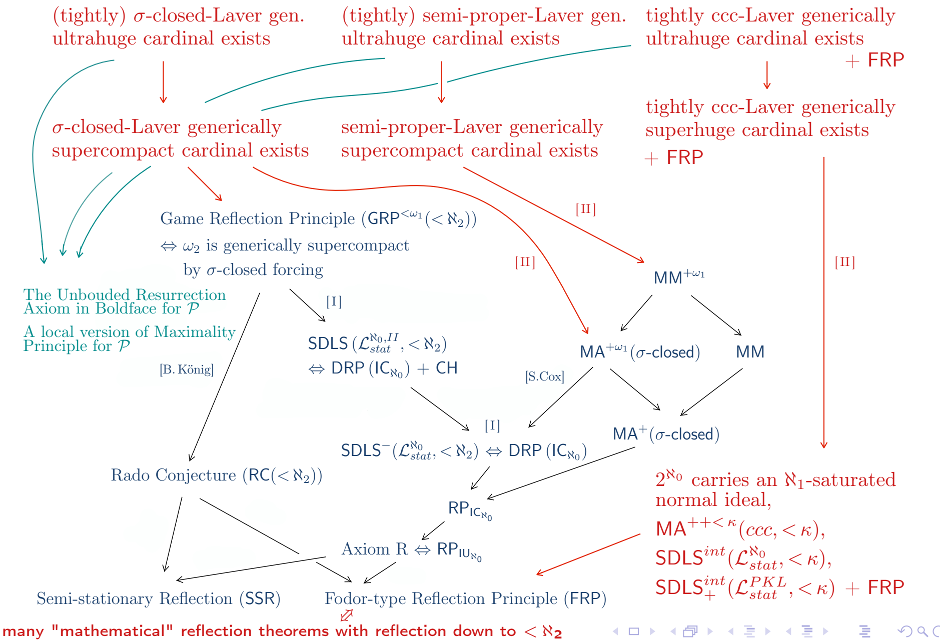
$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ " $V_{\mu^\bullet} \models \varphi(\check{a}_0, \ldots, \check{a}_{n-1})$", for all $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$",

we have $(V_{\mu^\bullet})^{\mathsf{V}} \models \varphi(a_0, \ldots, a_{n-1})$.

Proof. Let κ, φ, $a_0, \ldots, a_{n-1}$, $\mu^\bullet$, $\mathbb{P}$ as above. Let $\lambda > (\mu^\bullet)^{\mathsf{V}}$ be a limit ordinal. Then there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(V, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. ① $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, ② $j(\kappa) > \lambda$, ③ $\mathbb{P}$, $\mathbb{H}$, $(V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \in M$, and ④ $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j(\kappa)$.

- By the choice of λ and ① , we have $j(\lambda) > (\mu^\bullet)^M$. By ③ and ④ , we have $(V_{j(\lambda)})^M = (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]}$. Since $\mu^\bullet$ is a local definiton, it follows that $(\mu^\bullet)^M = (\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$. Thus, by the choice of $\mathbb{P}$, we have $M \models$ " $V_{\mu^\bullet} \models \varphi(a_0, \ldots, a_{n-1})$". Since $a_i = j(a_i)$ for $i < n$ by ① , it follows by the elementarity that $(V_{\mu^\bullet})^{\mathsf{V}} \models \varphi(a_0, \ldots, a_{n-1})$. □ (Theorem 16)

Tightly Laver-gen. ultrahugeness unifies everything



Further references

[Barbanel-DiPrisco-Tan] Julius B. Barbanel, Carlos A. Di Prisco, and It Ben Tan, Many-Times Huge and Superhuge Cardinals, The Journal of Symbolic Logic, Vol.49, No.1 (1984), 112–122.

[S.Cox] Sean Cox, The digaonal reflection principle, Proceedings of the American Mathematical Society, Vol.140, No.8 (2012), 2893-2902.

[S.F.-Juhász-et al.] S.F., István Juhász, Lajos Soukup, Zoltán Szentmiklóssy and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications, Vol.157, 8 (June 2010), 1415–1429.
https://fuchino.ddo.jp/papers/ssmL-erice-x.pdf

[S.F.-Sakai] S.F., and Hiroshi Sakai, Generically supercompact cardinals by forcing with chain conditions, RIMS Kôkûroku, No.2213, (2022), 94–111.
https://fuchino.ddo.jp/papers/RIMS2021-ccc-gen-supercompact-x.pdf

[S.F.-Sakai 2] S.F., and Hiroshi Sakai, The first-order definability of generic large cardinals, to appear.
https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf

Further references (2/2)

[Hamkins-Johnstone 1] Joel David Hamkins, and Thomas A. Johnstone, Resurrection axioms and uplifting cardinals, Archive for Mathematical Logic, Vol.53, Iss.3-4, (2014), 463–485.

[Hamkins-Johnstone 2] Joel David Hamkins, and Thomas A. Johnstone, Strongly uplifting cardinals and the boldface resurrection axioms, Archive for Mathematical Logic volume 56, (2017), 1115–1133.

[Higher-Inf] Akihiro Kanamori, The Higher Infinite, Springer-Verlag (1994/2003).

[B.König] Bernhard König, Generic compactness reformulated, Archive for Mathematical Logic 43, (2004), 311–326.

Thank you for your attention!
ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다

Σας ευχαριστώ για την προσοχή σας.

Dziękuję za uwagę.

Ich danke Ihnen für Ihre Aufmerksamkeit.

A Sketch of the Proof of Theorem 14.

- We prove Theorem 14. (3). The proof of (1) and (2) can be done similarly.
- Suppose that κ is ultrahuge and $f : \kappa \to V_\kappa$ is an ultrahuge Laver-function. In particular, this means:

For any set a and $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$, $j(f)(\kappa) = a$, $j(\kappa) > \lambda$, and ${}^{j(\kappa)}M, V_{j(\lambda)} \subseteq M$.

- Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a FS-iteration with

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \text{ is a ccc p.o.;”} \\ \mathbb{1}, & \text{otherwise.} \end{cases}$$

- We show that $\mathbb{P}_\kappa$ forces that κ is tightly ccc-Laver generically ultrahuge.
- Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. Suppose $\lambda > \kappa$ and $\mathbb{P}$ be a ccc p.o. in $\mathsf{V}[\mathbb{G}_\kappa]$. Let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{P}$.
- By Lemma 12 , we may assume that λ is inaccessible.
- Let $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$ be s.t. $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$, $j(\kappa) > \lambda$, and (*) ${}^{j(\kappa)}M, V_{j(\lambda+1)} \subseteq M$.

A Sketch of the Proof of Theorem 14. (2/3)

- ► $\mathbb{G}_\kappa$ is a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. $\lambda > \kappa$ and $\mathbb{P}$ is a ccc p.o. in $\mathsf{V}[\mathbb{G}_\kappa]$. $\underset{\sim}{\mathbb{P}}$ is a $\mathbb{P}_\kappa$-name of $\mathbb{P}$. λ is inaccessible.
- ▷ Let $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$ be s.t. $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$, $j(\kappa) > \lambda$, and (*) ${}^{j(\kappa)}M, V_{j(\lambda+1)} \subseteq M$.

- ► By elementarity, we have

  $M \models$" $j(\mathbb{P}_\kappa)$ is a FS-iteration of ccc p.o.s $\langle \mathbb{P}^*_\alpha, \underset{\sim}{\mathbb{Q}}^*_\beta : \alpha \leq j(\kappa), \beta < j(\kappa) \rangle$ with the book-keeping $j(f)$".

- ▷ Note that $\mathbb{P}^*_\alpha = \mathbb{P}_\alpha$ for all $\alpha \leq \kappa$, $\mathbb{P}_\kappa \in M$, and $\underset{\sim}{\mathbb{Q}}^*_\kappa = \underset{\sim}{\mathbb{P}}$.

  Thus, by the Factor Lemma

  $M[\mathbb{G}_\kappa] \models$" $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa$ is (forcing equivalent to) a FS-iteration of ccc p.o.s of length $j(\kappa)$ and its 0th iterand is $\mathbb{P}$".

- ▷ By the ccc of $\mathbb{P}_\kappa$ and (*), we have ${}^{\lambda}(M[\mathbb{G}_\kappa]) \subseteq M[\mathbb{G}_\kappa]$.
  In particular, ${}^{\omega}(M[\mathbb{G}_\kappa]) \subseteq M[\mathbb{G}_\kappa]$, and

  $\mathsf{V}[\mathbb{G}_\kappa] \models$" $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa$ is (forcing equivalent to) a FS-iteration of ccc p.o.s of length $j(\kappa)$ and its 0th iterand is $\mathbb{P}$".

- ▷ It follows that, in $\mathsf{V}[\mathbb{G}_\kappa]$, we have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*$ where $\mathsf{V}[\mathbb{G}_\kappa] \models\ \Vdash_{\mathbb{P}}$" $\underset{\sim}{\mathbb{Q}}^*$ is ccc".

A Sketch of the Proof of Theorem 14. (3/3)

▷ It follows that, in $V[\mathbb{G}_\kappa]$, we have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*$ where $V[\mathbb{G}_\kappa] \models \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}^*$ is ccc". (*) ${}^{j(\kappa)}M, V_{j(\lambda+1)} \subseteq M$.

- Let $\mathbb{H}$ be a $(V[\mathbb{G}_\kappa], j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa)$-geneneric filter:

▷ Note that $\mathbb{H}$ corresponds to a $(V[\mathbb{G}], \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*)$-generic filter, and $\mathbb{G}_\kappa * \mathbb{H}$ corresponds to a $(V, j(\mathbb{P}_\kappa))$-generic filter extending $\mathbb{G}_\kappa$.
I shall denote the latter also with $\mathbb{G} * \mathbb{H}$.

- Let $\tilde{j}$ be the "lifting" of j defined by

$$\tilde{j} : V[\mathbb{G}_\kappa] \to M[\mathbb{G}_\kappa * \mathbb{H}]; \quad \underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j(\underset{\sim}{a})[\mathbb{G}_\kappa * \mathbb{H}] \quad \text{for all } \mathbb{P}_\kappa\text{-name } \underset{\sim}{a}.$$

▷ Then we have $j \subseteq \tilde{j}$, $\tilde{j} : V[\mathbb{G}_\kappa] \xrightarrow{\prec}_\kappa M[\mathbb{G}_\kappa * \mathbb{H}]$, $\tilde{j}''\lambda = j''\lambda \in M \subseteq M[\mathbb{G}_\kappa * \mathbb{H}]$, $|j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa|^{V[\mathbb{G}_\kappa]} \leq |j(\mathbb{P}_\kappa)|^M = j(\kappa)$.

- $\mathbb{G}_\kappa * \mathbb{H}$ seen as a $(V, j(\mathbb{P}_\kappa))$-gen. filter has cardinality $j(\kappa) < j(\lambda)$ and it is $\in V_{j(\lambda)}$.

▷ Thus, there is a a $j(\mathbb{P}_\kappa * \underset{\sim}{\mathbb{Q}})$-name $\underset{\sim}{V}$ of $(V_{j(\lambda)})^{V[\mathbb{G}_\kappa * \mathbb{H}]}$ in $V_{j(\lambda)+1}$.

▷ It follows $(V_{j(\lambda)})^{V[\mathbb{G}_\kappa * \mathbb{H}]} = \underset{\sim}{V}[\mathbb{G}_\kappa * \mathbb{H}] \in M[\mathbb{G}_\kappa * \mathbb{H}]$.

- This shows that $V[\mathbb{G}_\kappa] \models$" κ is tightly ccc-Laver-gen. ultrahuge".

□ (Theorem 14. .)

Proof of Theorem 8.

- Suppose $A \subseteq \mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P} \in \mathcal{P}$. By tightly $\mathcal{P}$-Laver-gen. superhugeness of $\kappa_{\mathfrak{refl}}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a p.o. with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\tilde{\mathbb{H}}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with ① $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa_{\mathfrak{refl}}} M$, ② $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j(\kappa_{\mathfrak{refl}})$, ③ $\mathbb{P}$, $\mathbb{H} \in M$, and ④ $j''j(\kappa_{\mathfrak{refl}}) \in M$.

⑤ By ② , we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa_{\mathfrak{refl}})$.

▷ Since $crit(j) = \kappa_{\mathfrak{refl}}$, $j(a) = a$ for all $a \in (\mathcal{H}(\kappa_{\mathfrak{refl}}))^{\mathsf{V}}$.

Claim. $\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]} \subseteq M$ and hence $\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^M = \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}$.

$\vdash$ Suppose that $b \in \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}$ and let $c \subseteq j(\kappa_{\mathfrak{refl}})$ be a code of b. Let $\underset{\sim}{c}$ be a nice $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-name of c. By ②, $|\underset{\sim}{c}| \leq j(\kappa_{\mathfrak{refl}})$. By ④, it follows that $\underset{\sim}{c} \in M$. Thus $c \in M$ by ③ , and hence $b \in M$. $\dashv$

- Thus,

$$\underbrace{j \restriction \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}}_{= id_{\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}}} : (\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}, A, \in) \overset{\prec}{\rightarrow} (\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}, j(A), \in).$$

back

Proof of Theorem 7

- We prove Theorem 7., (1). (2) can be proved similarly.
- Assume that $\kappa > \aleph_1$ is $\mathcal{P}$-Laver-generically supercompact and elements of $\mathcal{P}$ are ccc.
- Suppose that $\mathbb{P} \in \mathcal{P}$, $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\,\mathcal{D}\,| < \kappa$, and $\mathcal{S}$ is a family fo $\mathbb{P}$-names s.t. $|\,\mathcal{S}\,| < \kappa$, and, for each $\underset{\sim}{S} \in \mathcal{S}$, there are $\omega < \eta_{\underset{\sim}{S}} \leq \theta_{\underset{\sim}{S}} < \kappa$ s.t. $\eta_{\underset{\sim}{S}}$ is regular, and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$".
- W.l.o.g., ① the underlying set of $\mathbb{P}$ is a cardinal $\lambda > \kappa$ and elements of $\mathcal{S}$ are nice names.
- Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and, for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. ② $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, ③ $j(\kappa) > \lambda$, ④ $\mathbb{P}, \mathbb{H} \in M$, and ⑤ $j''\lambda \in M$.
- Let $\mathbb{G}$ be the $\mathbb{P}$ part of $\mathbb{H}$. $\mathbb{G} \in M$ by ④ . $j''\mathbb{P} \subseteq j(\mathbb{P})$, and $j''\mathbb{P}$, $j \restriction \mathbb{P} \in M$ by the choice ① of $\mathbb{P}$, and ⑤ .

Proof of Theorem 7 (2/2)

- ▷ $j(\mathcal{D}) = \{j(D) : D \in \mathcal{D}\}$, and $j''D \subseteq j(D)$ for all $D \in \mathcal{D}$.
  $j(\mathcal{S}) = \{j(\underset{\sim}{S}) : \underset{\sim}{S} \in \mathcal{S}\}$, and $j''\underset{\sim}{S} \subseteq j(\underset{\sim}{S})$ for all $\underset{\sim}{S} \in \mathcal{S}$ and hence
  $\Vdash_{j(\mathbb{P})}$ "$j''\underset{\sim}{S} \subseteq j(\underset{\sim}{S}) \subseteq \mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$".
- ▶ Note that $j''\underset{\sim}{S}[j''\mathbb{G}] = \underset{\sim}{S}[\mathbb{G}]$ and
  $\underset{\sim}{S}[\mathbb{G}]$ is stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ (in $\mathsf{V}[\mathbb{G}]$ by genericity of $\mathbb{G}$, and hence also in M) for all $\underset{\sim}{S} \in \mathcal{S}$.
- ▷ Thus, in M, $j''\mathbb{G}$ generates a $j(\mathcal{D})$-generic filter on $j(\mathbb{P})$ which establishes the stationarity of interpretations of elements of $j(\mathcal{S})$.
- ▷ It follows that
  $M \models$" there is a $j(\mathcal{D})$-generic filter on $j(\mathbb{P})$ which establishes
  the stationarity of interpretations of elements of $j(\mathcal{S})$".
- ▷ By elementarity,
  $\mathsf{V} \models$" there is a $\mathcal{D}$-generic filter on $\mathbb{P}$ which establishes
  the stationarity of interpretations of all elements of $\mathcal{S}$".

back □ (Theorem 7)

A Sketch of the Proof of Theorem 5

- We prove Theorem 5.(3). The proof of (1) and (2) can be done similarly.
- Suppose that κ is supercompact and $f : \kappa \to V_\kappa$ is a supercompact Laver-function. In particular, this means:

For any set a and $\lambda > \kappa$, there are j, $M \subseteq V$ s.t. $j : V \overset{\preceq}{\to}_\kappa M$, $j(f)(\kappa) = a$, $j(\kappa) > \lambda$, and ${}^\lambda M \subseteq M$.

- Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a FS-iteration with

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \text{ is a ccc p.o.; ”} \\ \mathbb{1}, & \text{otherwise.} \end{cases}$$

- We show that $\mathbb{P}_\kappa$ forces that κ is tightly ccc-Laver generically supercompact. (the proof for “ccc-Laver gen. superhuge” etc. can be done similarly starting from a superhuge cardinal with superhuge Laver-function, etc.) The following is skipped sinsce we shall check it in the next talk.
- Let $\mathbb{G}_\kappa$ be a $(V, \mathbb{P}_\kappa)$-generic filter. Suppose $\lambda > \kappa$ and $\mathbb{P}$ is a ccc p.o. in $V[\mathbb{G}_\kappa]$. Let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{P}$.
- Let $j : V \overset{\preceq}{\to}_\kappa M$ be s.t. $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$, $j(\kappa) > \lambda$ and (*) ${}^\lambda M \subseteq M$.

A Sketch of the Proof of Theorem 5 (2/3)

- Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. Suppose $\lambda > \kappa$ and $\mathbb{P}$ is a ccc p.o. in $\mathsf{V}[\mathbb{G}_\kappa]$. Let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{P}$.
  - Let $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$ be s.t. $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$, $j(\kappa) > \lambda$ and (*) ${}^\lambda M \subseteq M$.
- By elementarity, we have

  $M \models$" $j(\mathbb{P}_\kappa)$ is a FS-iteration of ccc p.o.s $\langle \mathbb{P}^*_\alpha, \underset{\sim}{\mathbb{Q}}^*_\beta : \alpha \leq j(\kappa), \beta < j(\kappa) \rangle$ with the book-keeping $j(f)$".
  - Note that $\mathbb{P}^*_\alpha = \mathbb{P}_\alpha$ for all $\alpha \leq \kappa$, $\mathbb{P}_\kappa \in M$, and $\underset{\sim}{\mathbb{Q}}^*_\kappa = \underset{\sim}{\mathbb{P}}$.
  Thus, by the Factor Lemma

  $M[\mathbb{G}_\kappa] \models$" $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa$ is (forcing equivalent to) a FS-iteration of ccc p.o.s of length $j(\kappa)$ and its 0th iterand is $\mathbb{P}$".
  - By the ccc of $\mathbb{P}_\kappa$ and (*), we have ${}^\lambda(M[\mathbb{G}_\kappa]) \subseteq M[\mathbb{G}_\kappa]$.
  In particular, ${}^\omega(M[\mathbb{G}_\kappa]) \subseteq M[\mathbb{G}_\kappa]$, and

  $\mathsf{V}[\mathbb{G}_\kappa] \models$" $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa$ is (forcing equivalent to) a FS-iteration of ccc p.o.s of length $j(\kappa)$ and its 0th iterand is $\mathbb{P}$".

* This is the place where the corresponding proof of (2) needs the condition "${}^{j(\kappa)>}M \subseteq M$" to show the iteration is RCS-support of semi-proper forcing in $\mathsf{V}[\mathbb{G}_\kappa]$.
  - It follows that, in $\mathsf{V}[\mathbb{G}_\kappa]$, we have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*$ where $\mathsf{V}[\mathbb{G}_\kappa] \models \Vdash_{\mathbb{P}}$" $\underset{\sim}{\mathbb{Q}}^*$ is ccc".

A Sketch of the Proof of Theorem 5 (3/3)

▷ It follows that, in $V[\mathbb{G}_\kappa]$, we have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*$ where $V[\mathbb{G}_\kappa] \models \Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}}^*$ is ccc".

- Let $\mathbb{H}$ be a $(V[\mathbb{G}_\kappa], j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa)$-geneneric filter:

▷ Note that $\mathbb{H}$ corresponds to a $(V[\mathbb{G}], \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*)$-generic filter, and $\mathbb{G}_\kappa * \mathbb{H}$ corresponds to a $(V, j(\mathbb{P}_\kappa))$-generic filter extending $\mathbb{G}_\kappa$. I shall denote the latter also with $\mathbb{G} * \mathbb{H}$.

- Let $\tilde{j}$ be the "lifting" of j defined by

$$\tilde{j} : V[\mathbb{G}_\kappa] \to M[\mathbb{G}_\kappa * \mathbb{H}]; \quad \underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j(\underset{\sim}{a})[\mathbb{G}_\kappa * \mathbb{H}]$$

for all $\mathbb{P}_\kappa$-name $\underset{\sim}{a}$.

▷ Then we have $j \subseteq \tilde{j}$, $\tilde{j} : V[\mathbb{G}_\kappa] \overset{\prec}{\to}_\kappa M[\mathbb{G}_\kappa * \mathbb{H}]$, $\tilde{j}''\lambda = j''\lambda \in M \subseteq M[\mathbb{G}_\kappa * \mathbb{H}]$, $|j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa|^{V[\mathbb{G}_\kappa]} \leq |j(\mathbb{P}_\kappa)|^M = j(\kappa)$.

- This shows that $V[\mathbb{G}_\kappa] \models$ " κ is tightly ccc-Laver-gen. supercompact".

□ (Theorem 5.)

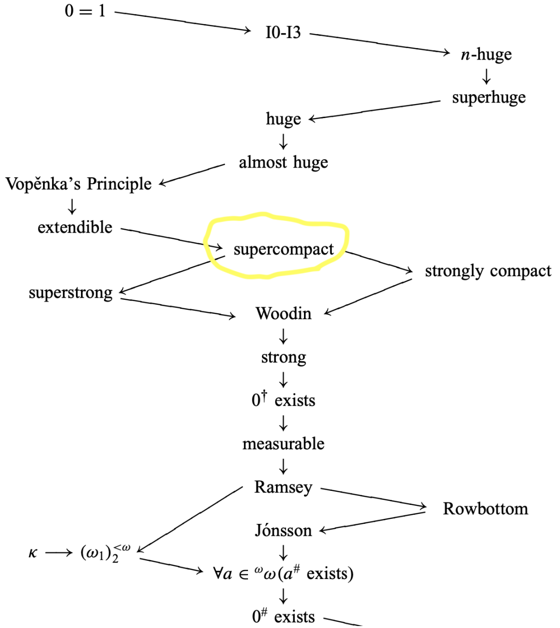
A ccc-gen. large cardinal is very large (but not necessarily a large cardinal)

Theorem 2ℵ. (Theorem 3.5 in [S.F.-Sakai] If κ is a ν-cc-gen. measurable cardinal for a $\nu < \kappa$, then κ is greatly weakly Mahlo.

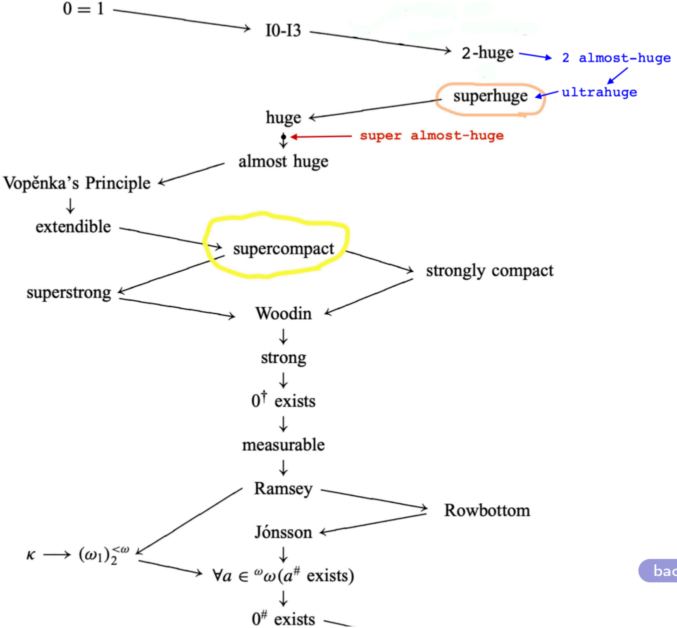
Theorem 3ℵ. (Theorem 3.7 in [S.F.-Sakai] Suppose that κ is a ν-cc-gen. measurable cardinal for some regular $\nu < \kappa$. Then κ is the stationary limit of ν-cc-gen. weakly compact cardinals below it.

back

The upper-half of the "Higher Infinite" [Higher-Inf]



The upper-half of the "Higher Infinite" [Higher-Inf]



back

Consistency of ∃ tightly ccc-Laver-gen. superhuge + FRP

- Suppose that κ is a superhuge cardinal.
- Then there are stationarily many supercompact cardinals below κ ([Barbanel-DiPrisco-Tan] , Theorem 7e.).
- ▷ Let κ_0 be one of them.
- Use κ_0 to force FRP (it is enough to force $MA^+(\sigma$-closed) — see ([S.F.-Juhász-et al.])
- Use κ to force "∃ tightly ccc-Laver-gen. superhuge" by FS-iteration of ccc p.o.s along with a superhuge Laver-function.
- FRP survives the second generic extension since FRP is preserved by ccc generic extensions (see ([S.F.-Juhász-et al.] , Theorem 3.4)

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[Higher-Inf] Proposition 26.11

- One of the strongest statements similar to Lemma 1 for a supercompact cardinal is the following:

Proposition 1ℵ. (Proposition 26.11 in [Higher-Inf])
If κ is 2^κ-supercompact, then there is a normal ultrafilter $\mathcal{U}$ over κ s.t.

$\{\alpha < \kappa \,:\, \alpha \text{ is superstrong}\} \in \mathcal{U}$.

- For cardinals $\kappa < \lambda$, κ is λ-supercompact if there is a $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, s.t. $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$.

- A cardinal κ is superstrong if there is a $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ with $V_{j(\kappa)} \subseteq M$.

A Sketch of the Proof of Proposition 1.

- Suppose that κ is a supercompact cardinal, $\mu < \kappa$ and $\mathcal{S} \subseteq [X]^\mu$ is stationary in $[X]^\mu$.
- Let $\lambda = |\,X\,|$. W.l.o.g., $\lambda \geq \kappa$.
- Let ① $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$ be s.t. ② $j(\kappa) > \lambda$, and ③ ${}^\lambda M \subseteq M$.
- We have $j''X \subseteq j(X)$ and $j(\mu) = \mu$ by ①. $j''X \in M$ by ③. Note $(X \cup \mathcal{S}, \mathcal{S}, \in) \cong (j''X \cup (j(\mathcal{S}) \cap [j''X]^\mu), j(\mathcal{S}) \cap [j''X]^\mu, \in)$.
- Thus we have:
  $M \models$ "$j(\mathcal{S}) \cap [j''X]^\mu$ is stationary in $[j''X]^\mu$".
- Hence
  $M \models$ "there is $Y \subseteq j(X)$, $|\,Y\,| = \lambda < j(\kappa)$ s.t. $j(\mathcal{S}) \cap [Y]^\mu$ is stationary".
- By elementarity ①, it follows that
  $\mathsf{V} \models$ "there is $Y \subseteq X$, $|\,Y\,| < \kappa$ s.t. $\mathcal{S} \cap [Y]^\mu$ is stationary".

- The last assertion of Proposition 1. is proved using the normal ultrafilter: $\mathcal{U} = \{A \subseteq [X]^{<\kappa} : j''X \in j(A)\}$ by showing $\{Y \in [X]^{<\kappa} : \mathcal{S} \cap [Y]^\mu \text{ is stationary}\} \in \mathcal{U}$. more details

A Sketch of the Proof of Proposition 1. – additional details

Notation and Definitions (2/2)

- $j : V \overset{\prec}{\to}_\kappa M \subseteq V$. ${}^{|X|}M \subseteq M$. $j(\kappa) > |X| \geq \kappa$.
  $\mathcal{U} = \{A \subseteq [X]^{<\kappa} : j''X \in j(A)\}$.

- $\mathcal{U}$ is $<\kappa$-complete: Suppose $A_\alpha \in \mathcal{U}$ for $\alpha < \mu < \kappa$. Then,
  $j''X \in \bigcap_{\alpha<\mu} j(A_\alpha) = \bigcup\{j(A_\alpha) : \alpha < j(\mu)\} = \bigcap j(\{A_\alpha : \alpha < \mu\}) = j(\bigcap\{A_\alpha : \alpha < \mu\})$. ▷ Thus $\bigcap\{A_\alpha : \alpha < \mu\} \in \mathcal{U}$.
- $\mathcal{U}$ is fine: Suppose $x \in X$. Then
  $j(\{a \in [X]^{<\kappa} : x \in a\}) = \{a \in [j(X)]^{<j(\kappa)} : j(x) \in a\} \ni j''X$.

▷ Thus $\{a \in [X]^{<\kappa} : x \in a\} \in \mathcal{U}$.

- Suppose $A_x \in \mathcal{U}$ for $x \in X$. ▷ Let $\langle \overline{A}_u : u \in j(X)\rangle := j(\langle A_x : x \in X\rangle)$.

▷ For any $u \in j''X$, there is $x \in X$ s.t. $u = j(x)$. Then we have
$j''X \in j(A_x) = \overline{A}_u$. This shows that $j''X \in j(\triangle_{x\in X} A_x)$. ▷ Thus $\triangle_{x\in X} A_x \in \mathcal{U}$.

- $\{Y \in [X]^{<\kappa} : \mathcal{S} \cap [Y]^\mu \text{ is stat.}\} \in \mathcal{U}$:

(*) $j(\{Y \in [X]^{<\kappa} : \mathcal{S} \cap [Y]^\mu \text{ is stat.}\}) = \{Y \in [j(X)]^{<j(\kappa)} : j(\mathcal{S}) \cap [Y]^\mu \text{ is stat.}\}$

▷ By the first part of the proof, $j''X \in$ right side of (*).
This proves the Claim above.

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Some Notation and Definitions

- For a set X, and a cardinal μ, $[X]^\mu := \{a \subseteq X : |a| = \mu\}$.

- Similarly: $[X]^{<\mu} := \{a \subseteq X : |a| < \mu\}$.
  $[X]^{<\mu}$ is sometimes also denoted by $\mathcal{P}_\mu(X)$.

- $\mathcal{C} \subseteq [X]^\mu$ is club (closed unbounded) in $[X]^\mu$ if ① for a $\subseteq$-chain $C \in [\mathcal{C}]^{\leq\mu}$, $\bigcup C \in \mathcal{C}$; and ② for any $a \in [X]^\mu$, there is $c \in \mathcal{C}$ s.t. $a \subseteq c$.

- $\mathcal{S} \subseteq [X]^\mu$ is stationary in $[X]^\mu$ if $\mathcal{S} \cap \mathcal{C} \neq \emptyset$ for all club $\mathcal{C} \subseteq [X]^\mu$.

- Clubness and stationarity of subsets of $[X]^{<\mu}$ (μ regular) is defined similarly.

Back

Some Notation and Definitions (2/2)

- An ultrafilter $\mathcal{U}$ over $[X]^{<\kappa}$ for regular κ is normal if ① $\mathcal{U}$ is $<\kappa$-complete (i.e. for any $\mathcal{S} \in [\mathcal{U}]^{<\kappa}$, $\bigcap \mathcal{S} \in \mathcal{U}$), ② $\mathcal{U}$ is fine (i.e. for any $x \in X$, $\{a \in [X]^{<\kappa} : x \in a\} \in \mathcal{U}$), and ③ for any $U_x \in \mathcal{U}$ for $x \in X$, $\triangle_{x \in X} U_x = \{a \in [X]^{<\kappa} : a \in U_x \text{ for all } x \in a\} \in \mathcal{U}$.

Lemma 4ℵ. Suppose that $\mathcal{U}$ is a normal ultrafilter over $[X]^{<\kappa}$. Then

(1) For any club $C \subseteq [X]^{<\kappa}$ we have $C \in \mathcal{U}$.
(2) Any $S \in \mathcal{U}$ is stationary subset of $[X]^{<\kappa}$.

Proof. (1): Suppose that $\mathcal{C} \subseteq [X]^{<\kappa}$ is a club. For each $x \in X$, let $c_x \in C$ be s.t. $x \in c_x$ (possible since C is a club).

- ▷ Let $U_x := \{a \in [X]^{<\kappa} : c_x \subseteq A\}$. $U_x \in \mathcal{U}$ by ②. Let $C_0 := \triangle_{x \in X} U_x$.
- $C_0 \in \mathcal{U}$ by ③ . $C_0 \subseteq C$:
- ▷ $a \in C_0 \;\Rightarrow\; c_x \subseteq a$ for all $x \in a \;\Rightarrow\; a = \bigcup_{x \in a} c_x \in C$ since C is club.
- Since $\mathcal{U}$ is a filter, it follows that $C \in \mathcal{U}$.

(2): follows from (1). □ (Lemma 4ℵ)