Resurrection and Maximality under a/the tightly Laver-generically ultrahuge cardinal — additional slides

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* the following are slides prepared in addition to the main slides:
https://fuchino.ddo.jp/slides/kobe2023-05-29-pf.pdf

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Outline

- ▷ References
- ▷ Outline

[The upper-half of the "Higher Infinite"] — chart given in the last talk

- ▷ Consistency strength of super almost-huge cardinal

[The upper-half of the "Higher Infinite"] — updated chart

- ▷ Irreversibility of some implications

- ▷ Maximality Principle
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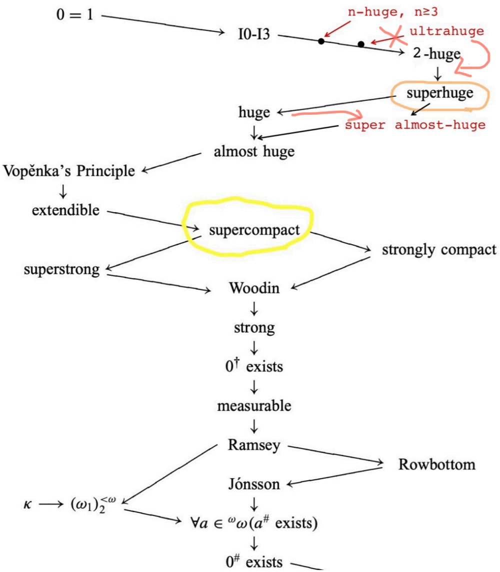
The upper-half of the "Higher Infinite" [Higher-Inf]

- Chart given in the last talk:



Consistency strength of super almost-huge cardinal

- [Theorem 3] is what I want to establish below.
- For cardinals $\kappa \le \lambda$ and a sequence $\vec{\mathcal{U}} = \langle \mathcal{U}_\gamma : \kappa \le \gamma < \lambda \rangle$ s.t. $\mathcal{U}_\gamma$ is a normal ultrafilter over $\mathcal{P}_\kappa(\gamma)$ for all $\kappa \le \gamma < \lambda$, we say that $\vec{\mathcal{U}}$ is coherent if $\mathcal{U}_\gamma = \mathcal{U}_\delta | \gamma := \{\{a \cap \gamma : a \in A\} : A \in \mathcal{U}_\delta\}$ for all $\kappa \le \gamma \le \delta < \lambda$,.
- ▷ For a coherent sequence of normal ultrafilters $\vec{\mathcal{U}} = \langle \mathcal{U}_\gamma : \kappa \le \gamma < \lambda \rangle$, We let $j_\gamma : \mathsf{V} \xrightarrow{\preceq}_\kappa M_\gamma \cong Ult(\mathsf{V}, U_\gamma)$ be the standard embedding, and, for $\kappa \le \gamma \le \delta < \lambda$, we define $k_{\gamma,\delta} : M_\gamma \xrightarrow{\preceq} M_\delta$ by $k_{\gamma,\delta}([f]_{\mathcal{U}_\gamma}) := [\langle f(x \cap \gamma) : x \in \mathcal{P}_\kappa(\delta) \rangle]_{\mathcal{U}_\delta}$.
- ▷ Then we have $j_\delta = k_{\gamma\delta} \circ j_\gamma$.

Theorem 1. ([Higher-Inf], Theorem 24.11 reformulated) For a cardinal number κ and inaccessible $\lambda > \kappa$ the following are equivalent:

(a) κ is a almost-huge cardinal with almost-huge elementary embedding j with the target $j(\kappa) = \lambda$.

(b) There is a coherent sequence $\langle \mathcal{U}_\gamma : \kappa \le \gamma < \lambda \rangle$ of normal ultrafilters s.t.

① for all $\kappa \le \gamma < \lambda$ and α with $\gamma \le \alpha < j_\gamma(\kappa)$, there is $\gamma \le \delta < \lambda$ s.t. $k_{\gamma,\delta}(\alpha) = \delta$.

Lemma 2. If κ is an (almost) huge cardinal and ② $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$ is a(n almost) huge elementary embedding. Thus, in particular,
③ ${}^{j(\kappa)>}M \subseteq M$. Then (1) $j(\kappa)$ is inaccessible.
(2) $\{\alpha < \kappa : \alpha$ is measurable$\}$ is normal measure 1 subset of κ.
(3) $M \models$ "$\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ is stationary in $j(\kappa)$".
(4) $\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ is cofinal in $j(\kappa)$.

Proof. (1): Since κ is inaccessible. $M \models$ "$j(\kappa)$ is inaccessible" by elementarity ② . By ③ , it follows that $j(\kappa)$ is really inaccessible.

(2): κ is measurable and an ultrafilter witnessing this is an element of M by ③ and (1). Thus $M \models$ "κ is measurable".
$\mathcal{U} := \{A \subseteq \kappa : \kappa \in j(A)\}$ is a normal ultrafilter over κ and $\{\alpha < \kappa : \alpha$ is measurable$\} \in \mathcal{U}$. (3): By (2), $\{\alpha < \kappa : \alpha$ is measurable$\}$ is a stationary subset of κ. By elementarity ② , it follows that $M \models$ "$\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ is stationary $\subseteq j(\kappa)$". (4): follows from (3) and ③ . $\square$ (Lemma 2)

Consistency strength of super almost-huge cardinal (3/3)

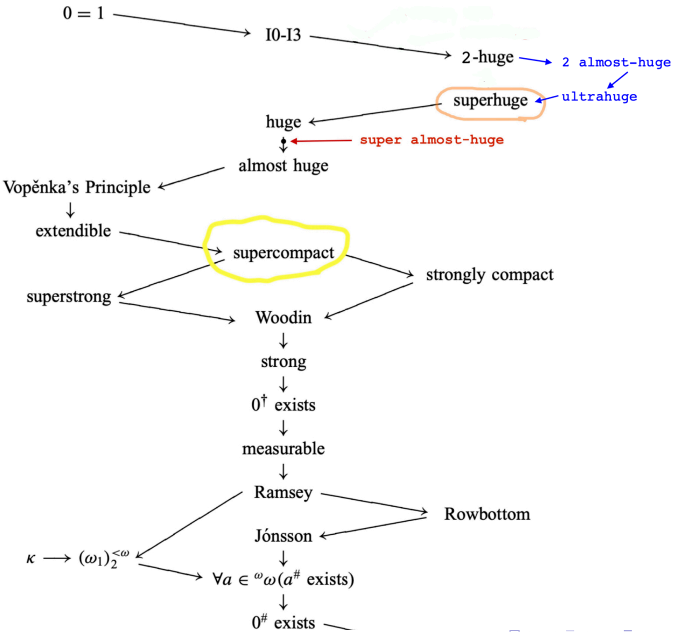
Theorem 3. Suppose that κ is huge. Then, $\{\alpha < \kappa : V_\kappa \models$ "α is super almost-huge"$\}$ is a normal measure 1 subset of κ.

see the extended chart

Proof. Let $j : V \xrightarrow{\preceq}_\kappa M$ be a huge elementary embedding, so that we have ④ ${}^{j(\kappa)}M \subseteq M$.

- For $\kappa \leq \gamma < j(\kappa)$, let $\mathcal{U}_\gamma := \{A \subseteq \mathcal{P}_\kappa(\gamma) : j''\gamma \in j(A)\}$.
  - Then $\vec{\mathcal{U}} := \langle \mathcal{U}_\gamma : \kappa \leq \gamma < j(\kappa)\rangle \in M$ by ④, and $\vec{\mathcal{U}} \models$ ① (see the proof of [Higher-Inf], Theorem 24.11).
  - Since ① is a closure property, M knows that there are club many $\alpha < j(\kappa)$ s.t. $\langle \mathcal{U}_\gamma : \kappa \leq \gamma < \alpha\rangle \models$ ① .
  - By Lemma 2, (2), M thinks that there are stationarily many $\alpha < \kappa$ which are inaccessible (actually even measurable!). Thus
- $M \models$ "there are stat. many inaccsessible $\alpha < j(\kappa)$ s.t. $\langle \mathcal{U}_\gamma : \kappa \leq \gamma < \alpha\rangle \models$ ①"
- By Theorem 1, ⑤ $M \models$ "$V_{j(\kappa)} \models \kappa$ is super almost-huge".
- $\mathcal{U} := \{A \subseteq \kappa : \kappa \in j(A)\}$ is a normal ultrafilter over κ.
- By ⑤, $\{\alpha < \kappa : V_\kappa \models$ "α is super almost-huge"$\} \in \mathcal{U}$. □ (Theorem 3)

The upper-half of the "Higher Infinite" [Higher-Inf]



Irreversibility of some implications

Proposition 4. Suppose that $\mathcal{P}$ = all σ-closed p.o.s or $\mathcal{P}$ = all ccc p.o.s. Then tightly $\mathcal{P}$-Laver gen. supercompactness of κ does not necessarily imply the $\mathcal{P}$-gen. ultrahugeness of κ.

▷ For the proof of Proposition 4, we use the following:

Lemma 5. Suppose that κ is $\mathcal{P}$-gen. ultrahuge for an arbitrary class $\mathcal{P}$ of p.o.s. If there is an inaccessible $\lambda_0 > \kappa$ then there are cofinally many inaccessible in V.

Proof of Lemma 5: Let $\lambda > \lambda_0$ be an arbitrary cardinal. Then there is $\mathbb{P} \in \mathcal{P}$ s.t., for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. ⑥ $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, ⑦ $j(\kappa) > \lambda$, and ⑧ $(V_{j(\lambda)})^{\mathsf{V}[\mathbb{G}]} \in M$.

- ▶ By ⑦ and elementarity ⑥ , we have $j(\lambda_0) > \lambda$.
- ▶ By elementarity ⑥ , $M \models$“ $j(\lambda_0)$ is inaccessible”.
- ▷ By ⑧ , $V[\mathbb{G}] \models$“ $j(\lambda_0)$ is inaccessible”, and hence $V \models$“ $j(\lambda_0)$ is inaccessible”.

□ (Lemma 5)

Irreversibility of some implications (2/2)

Proposition 4. Suppose that $\mathcal{P}$ = all σ-closed p.o.s or $\mathcal{P}$ = all ccc p.o.s. Then tightly $\mathcal{P}$-Laver gen. supercompactness of κ does not necessarily imply the $\mathcal{P}$-gen. ultrahugeness of κ.

Lemma 5. Suppose that κ is $\mathcal{P}$-gen. ultrahuge for an arbitrary class $\mathcal{P}$ of p.o.s. If there is an inaccessible $\lambda_0 > \kappa$ then there are cofinally many inaccessible in V.

Proof of Proposition 4: Suppose that κ is a supercompact cardinal and $\lambda_0 > \kappa$ is an inaccessible cardinal.

- ▶ We may assume that λ_0 is the largest inaccessible cardinal: If there is inaccessible cardinal larger than λ_0, then let λ_1 be the least such inaccessible cardinal. Then, in V_{λ_1}, λ_0 is the largest inaccessible cardinal and κ is supercompact.
- ▷ $V_{\lambda_1} \models$" κ is supercompact" can be seen using the characterization of supercompactness in terms of ultrafilters.
- ▶ By Theorem 5 (in the main slides), (a), (c), there is a po $\mathbb{P}$ of size κ s.t., for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] \models$" κ is tightly $\mathbb{P}$-Laver gen. supercompact".
- ▶ $\mathsf{V}[\mathbb{G}] \models$" λ_0 is the largest inaccessible caridnal".
- ▷ By Lemma 5 above, it follows that $\mathsf{V}[\mathbb{G}] \models$" κ is not $\mathcal{P}$-gen. ultrahuge".

□ (Proposition 4)

Maximality Principle

- Maximality Principle (MP) in its non parameterized form is formulated in an infinite set of formulas asserting that all buttons are already pushed. I.e., for any $\mathcal{L}_\varepsilon$-sentence φ, if, for a p.o. $\mathbb{P}$,

(*) $\Vdash_{\mathbb{Q}}$ "φ" for all $\mathbb{Q}$ with $\mathbb{P} \lessdot \mathbb{Q}$,

then φ holds.

- If (*) holds then we shall say that φ is a button with the push $\mathbb{P}$.

Proposition 6. MP implies $\mathsf{V} \neq \mathsf{L}$.

- For an $\mathcal{L}_\varepsilon$-sentence φ let

  $mp_\varphi \;:\leftrightarrow\; \exists P\,(P \text{ is a p.o.} \wedge \forall Q(P \lessdot Q \rightarrow \Vdash_Q \text{“}\varphi\text{”})) \rightarrow \varphi.$

- Formally we define: $\mathsf{MP} := \{mp_\varphi : \varphi \text{ is an } \mathcal{L}_\varepsilon\text{-sentence}\}.$

Maximality Principle (2/3)

Lemma 7. Suppose that φ is an $\mathcal{L}_\varepsilon$-sentence. If ZFC is consistent, then so is ZFC $+$ mp_φ.

Proof. Suppose otherwise. Then we have ⑨ ZFC $\vdash \neg mp_\varphi$.

- Note that

⑩ $\neg mp_\varphi \;\leftrightarrow\; \exists P\,(P \text{ is a p.o.} \wedge \forall Q(P \lessdot Q \rightarrow \Vdash_Q \text{“}\varphi\text{”})) \wedge \neg\varphi$.

- ▷ In ZFC, let $\mathbb{P}$ be a p.o. as above. Then $\Vdash_{\mathbb{P}} \text{“}\varphi\text{”}$.
- ▷ On the other hand, since $\Vdash_{\mathbb{P}} \text{“}\psi\text{”}$ for all $\psi \in$ ZFC and by ⑨ , ⑩ , we have $\Vdash_{\mathbb{P}} \text{“}\neg\varphi\text{”}$ which is equivalent to $\neg \Vdash_{\mathbb{P}} \text{“}\varphi\text{”}$.
- Thus we obtained a proof of contradiction from ZFC. This is a contradiction to our assumption. □ (Lemma 7.)

Lemma 8. For any $\mathcal{L}_\varepsilon$-sentences $\varphi_0, \ldots, \varphi_{n-1}$, we have ZFC $\vdash (mp_{\varphi_0} \wedge \cdots \wedge mp_{\varphi_{n-1}}) \;\leftrightarrow\; mp_{\varphi_0 \wedge \cdots \wedge \varphi_{n-1}}$.

Proof. If $\mathbb{P}_0, \ldots, \mathbb{P}_{n-1}$ are pushes of the buttons $\varphi_0, \ldots, \varphi_{n-1}$ resp., then $\mathbb{P}_0 \times \cdots \times \mathbb{P}_{n-1}$ is a push for $\varphi_0 \wedge \cdots \wedge \varphi_{n-1}$. □ (Lemma 8)

Maximality Principle (3/3)

Lemma 7. Suppose that φ is an $\mathcal{L}_\varepsilon$-sentence. If ZFC is consistent, then so is ZFC $+$ mp_φ. $\square$

Lemma 8. For any $\mathcal{L}_\varepsilon$-sentences $\varphi_0, \ldots, \varphi_{n-1}$, we have
ZFC $\vdash (mp_{\varphi_0} \wedge \cdots \wedge mp_{\varphi_{n-1}}) \leftrightarrow mp_{\varphi_0 \wedge \cdots \wedge \varphi_{n-1}}$. $\square$

Theorem 9. (Hamkins,[Hamkins]) If ZFC is consistent, then so is ZFC $+$ MP.

Proof. By Compactness Theorem, Lemma 7 and Lemma 8.

$\square$ (Theorem 9)

- The same proof also shows the following:

Theorem 10. Suppose that "x-large cardinal" is a notion of large cardinal s.t. "κ is an x-large cardinal" is preserved by set-forcing of size $< \kappa$. If ZFC $+$ "there are class many x-large cardinals" is consistent, then so is ZFC $+$ MP $+$ "there are class many x-large cardinals". $\square$

Maximality Principle is preserved by set-forcing

Theorem 11. (Hamkins, [Hamkins])
MP is preserved by any set-generic extension.

Proof. The theorem follows immediately from the following Lemma.

□ (Theorem 11)

Lemma 12. MP is equivalent to $\{mp^+_\varphi : \varphi$ is a $\mathcal{L}_\varepsilon$-sentence$\}$

where

$$mp^+_\varphi \;:\leftrightarrow\; \exists P\,(P \text{ is a p.o.} \wedge \forall Q(P \lessdot Q \to \Vdash_Q \text{``}\varphi\text{''}))$$
$$\to \forall R(R \text{ is a p.o. } \to \Vdash_R \text{``}\varphi\text{''}).$$

Proof. $\Leftarrow$: is trivial.

$\Rightarrow$: Write $\Box\varphi$ for $\forall R(R$ is a p.o. $\to \Vdash_R$ "φ").

▷ We have $\Box\varphi \;\leftrightarrow\; \Box\Box\varphi$. Thus $mp_{\Box\varphi}$ is equivalent to mp^+_φ.

□ (Lemma 12)

A normal notion of normal large cardinal

- A kind of inverse of Theorem 10 also holds:

Theorem 13. Suppose that MP holds. If "x-large cardinal" is a notion of large cardinal s.t. ① "κ is an x-large cardinal" implies that κ is inaccessible; ② "κ is an x-large cardinal" can not be destroyed by forcing of size $< \kappa$; ③ no new x-large cardinal is created by set-forcing.
If there is an x-large cardinal, then there are cofinally many x-large cardinals in V.

Proof. Suppose otherwise. Let κ_0 be a x-large cardinal, and $\kappa_1 > \kappa_0$ be a cardinal above which there are no x-large cardinals.

- Let $\mathbb{P}$ be a p.o. which collapses κ_1 to, say cardinality ω_1, and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. Then by ① and ②, there is no x-large cardinal in $\mathsf{V}[\mathbb{G}]$. Also there is no x-large cardinal in any further generic extention by ③.
- By MP it follows that there is no x-large cardinal in V but this is a contradiction to the assumption of the theorem. $\square$ (Theorem 13.)

A normal notion of normal large cardinal (2/2)

- We shall say a notion of large cardinal (call this notion "x-large cardinal") normal if ① "κ is an x-large cardinal" implies that κ is inaccessible.

② "κ is an x-large cardinal" cannot be destroyed by a forcing of size $< \kappa$.

③ No new x-large cardinal can be created by small set-forcing.

④ ZFC + "there are unboundedly many x-large cardinals" is consistent.

- Most of the known notions of large cardinal are normal in the sense above under the assumption of the consistency of the existence of a sufficiently large cardinal.

Example 14. The notion of super almost-huge cardinal is normal under the consistency of ZFC + "there is a huge cardinal" (Theorem 3).

- A normal notion of large cardinal "x-large cardinal" is suspiciously normal if "small" in ③ is dropped. The notion of "x-large cardinal" in Theorem 13 is rather suspiciously normal.

Maximality Principle is independent over Laver-ge. large cardinal

Theorem 10 reformulated. Suppose that "x-large cardinal" is a normal notion of large cardinal. Then
ZFC + MP + "there are class many x-large cardinals" is consistent.

Theorem 11. (Hamkins, [Hamkins])
MP is preserved by any set-generic extension.

Theorem 13 reformulated. Assume that MP holds. If "x-large cardinal" is a normal notion of of large cardinal and there is at least one x-large cardinal, then there are cofinally many x-large cardinals in V.

Theorem 15. Suppose that $\mathcal{P}$ is an iterable class of p.o.s, and "x-large cardinal" is a normal notion of large cardinal s.t. its (tightly) Laver-generic version is well-defined and can be forced starting from an x-large cardinal κ by a set forcing of small size, then MP is consistent with ZFC + " there exists a (tightly) $\mathcal{P}$-gen. Laver-gen. x-large cardinal".

If, in addition, "there exist y-large cardinals above an x-large cardinal but only boundedly many" is consistent for a suspiciously normal notion of large cardinal "y-large cardinal", then MP is independent over ZFC + " there exists a (tightly) $\mathcal{P}$-gen. Laver-gen. x-large cardinal".

Proof. By Theorem 10, ZFC + MP + "there are class many x-large cardinals" is consistent.

Maximality Principle is independent over Laver-ge. large cardinal (2/2)

▷ Starting from a model of this theory, if we force the existence of (tightly) $\mathcal{P}$-Laver-gen. x-large cardinal by a set-forcing then MP survives in the generic extension by Theorem 11.

▷ This shows the consistence of ZFC + MP + "there is a (tightly) $\mathcal{P}$-Laver gen. x-large cardinal".

▶ For the second-half of the theorem, we start from a model with an x-large cardinal κ_0 and with at least one but only many y-cardinals above κ_0.

▷ Working in such a model V, Force the existence of (tightly) $\mathcal{P}$-Laver gen. x-large cardinal using κ_0.

▷ Let $V[\mathbb{G}]$ be the generic extension. By the properties ② and ③ of normality there are y-large cardinals in $V[\mathbb{G}]$ but they are bounded.

▷ By Theorem 13, it follows that $V[\mathbb{G}] \models \neg MP$. □ (Theorem 15)

Corollary 16. Suppose that $\mathcal{P}$ is an iterable class of p.o.s for which a forcing construction of $\mathcal{P}$-Laver gen. supercompact cardinal like the one in Theorem 5, (1) or (3) in the main slides is available. Then MP is independent over ZFC + "there is a $\mathcal{P}$-Laver gen. supercompact cardinal".

Proof. Use "inaccessible" as "y-large cardinal" in Theorem 15. □ (Corollary 16)

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Thank you for your attention!

ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다

Σας ευχαριστώ για την προσοχή σας.

Dziękuję za uwagę.

Ich danke Ihnen für Ihre Aufmerksamkeit.