Geology of set-theoretic Multiverse

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▶ **2025 年 12 月 17 日 (14:40～15:20 JST) : RIMS set theory workshop 2025**

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Set-theoretic Multiverse

- In modern set theory, models of set theory are often constructed starting from a "universe" by taking its generic extension, or by taking its inner model, or else by some combination of these operations. The attitude looking at the collection of all these models as the (possibly ultimate) cosmos of mathematics is called "set-theoretic multiverse".
  - Usually we take a countable transitive model of set theory as the initial universe M so that we can actually construct the M-generic set $\mathbb{G}$ for a p.o. $\mathbb{P} \in M$ (we also say $\mathbb{G}$ is an $(M, \mathbb{P})$-generic set in this context).
  - However we also often talk about generic extensions of the (real) universe V.
    — This is merely a sort of modus operandi which actually makes no sense, since V, being the class of all sets, can not afford any sets outside it.
    — However we know how to handle this apparent paradox (see e.g. [1] ,[2]).

[1] Kenneth Kunen, Set Theory: An Introduction to Independence Proofs (1980).
[2] S.F., Iterated forcing, Lecture note (2018).

Set-theoretic Multiverse (2/4)

- For the purpose of the following discussion, we consider an assertion like "For $(\mathrm{V}, \mathbb{P})$-generic $\mathbb{G}$, $\mathrm{V}[\mathbb{G}] \models \varphi$ holds" simply as an abbreviation of $\Vdash_{\mathbb{P}} $ "φ".
- For a (set or class) model M of ZFC, $N \subseteq M$ is a ground of M if N is an inner model of ZFC in M (i.e. $N \models$ ZFC, N is transitive and $\mathrm{On}^N = \mathrm{On}^M$) and there is a p.o. $\mathbb{P} \in N$ and $(N, \mathbb{P})$-generic $\mathbb{G} \in M$ s.t. $M = N[\mathbb{G}]$.

Theorem 1. (Woodin, Laver, independently, see e.g. [3]) Each ground N in M is uniformly definable using a parameter from N.

- Hamkins called the study of grounds of the universe V and also other (definable) inner models of V, more generally, the Set-theoretic Geology.
- Note that, by Theorem 1., we can talk e.g. about a set-indexed family $\mathcal{F}$ of grounds in V.

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[3] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz, Set-theoretic geology, Ann.of P. and Appl. Logic Vol.166, (2015), 464l–501.

Set-theoretic Multiverse (3/4)

Theorem 2. (Usuba [4]) For any set-indexed family $\mathcal{F}$ of grounds, there is a ground W (of the universe) s.t. W is a lower bound of all members of $\mathcal{F}$ (w.r.t. $\subseteq$).

Hamkins: "To my way of thinking, ..."

- In the following, when we are formulating things in a semantic narration, we call the universe in which we are "living" and from which we start the discussion of the set-theoretic multiverse, the initial universe. The initial universe can be the real universe V but it can also be a transitive (or even, possibly set) model M of ZFC or a model of some large enough finite fragment of ZFC, which is chosen at the start of the argument.

Corollary 3. If a model N is attained form the initial universe M by application of the operations of taking a generic extension and taking a ground, it can be represented as a generic extension of a ground of M.

[4] Toshimichi Usuba, The downward directed grounds hypothesis and very large cardinals, J. of Math. Logic Vol.17, No.02, (2017).

Set-theoretic Multiverse (3/4)

- Let M be a countable transitive model of ZFC, and let

  $$\mathcal{MV}_0^M := \{N : N \text{ is a generic extension of a ground of } M\}.$$

  By Corollary 3, $\mathcal{MV}_0^M$ is closed under the operations of taking a generic extension, and taking a ground.

- $\mathcal{MV}_0^M$ could be seen as a miniature model of set-theoretic multiverse in which we could perform "Gedankenexperimenten" about the "real" multiverse.

- However, there is one serious problem with $\mathcal{MV}_0^M$:

  $\mathcal{MV}_0^M$ does not have the amalgamation property.

Proposition 4. (Woodin, (see Hamkins [5])) For a countable transitive model M of ZFC and $\mathbb{P} \in M$ with $M \models \mathbb{P} = \mathrm{Fn}(\omega, \omega)$, there are $(M, \mathbb{P})$-generic $\mathbb{G}_0$, $\mathbb{G}_1$ such that $M[\mathbb{G}_0]$ and $M[\mathbb{G}_1]$ do not have any common extension of the form $M[\mathbb{G}]$.

Proof. Let $f \in {}^{\omega}2$ be s.t. f codes a bijection from ω to On^M.

- Let D_n, $n \in \omega$ enumerate open dense subsets of $\mathbb{P}$ in M.
- $\mathbb{p}^0_n$, $\mathbb{p}^1_n \in {}^{\omega>}\omega$ for $n \in \omega$ be two decreasing sequences in $\mathbb{P}$ s.t.
  - ▷ $\mathbb{p}^0_0(0) = f(0)$, $\mathbb{p}^0_0(1) = 0$, $\mathbb{p}^0_0 \in D_0$; ▷ $\mathbb{p}^1_0(0) = \mathrm{dom}(\mathbb{p}^0_0)$, $\mathbb{p}^1_0 \in D_0$;
  - ▷ $\mathbb{p}^0_{n+1}(\mathrm{dom}(\mathbb{p}^0_n)) = f(n+1)$, $\mathbb{p}^0_{n+1}(\mathrm{dom}(\mathbb{p}^0_n)+1) = \mathrm{dom}(\mathbb{p}^1_n), \mathbb{p}^0_{n+1} \in D_{n+1}$;
  - ▷ $\mathbb{p}^1_{n+1}(\mathrm{dom}(\mathbb{p}^1_n)) = \mathrm{dom}(\mathbb{p}^0_{n+1})$; $\mathbb{p}^1_{n+1} \in D_{n+1}$.
- Let $\mathbb{G}_i$ be the filter on $\mathrm{Fn}(\omega, \omega)$ generated from $\{\mathbb{p}^i_n : n \in \omega\}$ for $i \in 2$. Then $\mathbb{G}_i$ are $(M, \mathbb{P})$-generic.
- f can be reconstructed from $\mathbb{G}_0$ and $\mathbb{G}_1$. Hence there can be no $M[\mathbb{G}]$ with $M[\mathbb{G}_0]$, $M[\mathbb{G}_1] \subseteq M[\mathbb{G}]$. ⧉ (Proposition 4.)

[5] Joel D. Hamkins, Upward closure and amalgamation in the generic multiverse of a countable model of set theory, 数理解析研究所講究録 (Rims Kôkyû-roku) 第 1988 巻, (2016), 17–30.

Steel's multiverse

- Note that $M[\mathbb{G}_0]$, $M[\mathbb{G}_1]$ cannot be amalgamated into a model of the form $M[\mathbb{G}]$ even if the subset relation is replaced by elementary embedding.
- Steel's model of multiverse solves the problem of "multiverse $\mathcal{MV}_0^M$ without amalgamation property"[6], [7].
- Let M be an arbitrary countable transitive model of ZFC and let $\mathbb{G}$ be a $(M, \mathrm{Col}(\omega, <\mathsf{On}^M))$-generic filter (or $(M, \mathrm{Col}(\omega, \mathsf{On}^M))$-generic filter, in Kanamori's notation).
- Note that $\mathrm{Col}(\omega, <\mathsf{On}^M)$ is a class forcing in M and all ordinals in M are collapsed and become countable. In particular, $M[\mathbb{G}]$ is not a model of ZFC.

  $\mathcal{MV}_{\mathsf{ST}}^{M,\mathbb{G}} := \{N : N$ is a ground of $M[\mathbb{G} \restriction \alpha]$ for some $\alpha \in \mathsf{On}^M\}$.

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[6] John. R. Steel, Gödel's program, in: J. Kennedy (ed.), Interpreting Gödel: Critical Essays. Cambridge, UK: Cambridge University Press (2014).

[7] ____________, Generically invariant set theory, in: S. Arbeiter, and J. Kennedy (eds.), The Philosophy of Penelope Maddy, Springer (2024).

Steel's multiverse (2/2)

- It is easy to see that $\mathcal{MV}^{M,\mathbb{G}}_{\mathsf{ST}}$ satisfies the amalgamation property (w.r.t. "$\subseteq$"). Actually, Steel introduced $\mathcal{MV}^{M,\mathbb{G}}_{\mathsf{ST}}$ as a model of the following theory of multiverse MV ([6] , [7]): Let $\mathcal{L}_{\mathsf{MV}}$ be the language $\{\in, \mathcal{S}, \mathcal{W}\}$ ($\mathcal{S}$ and $\mathcal{W}$ are unary predicates).

- Axioms of MV are:

(1) $\forall x\,(\mathcal{S}(x) \vee \mathcal{W}(x))$, $\forall x\,(\mathcal{S}(x) \rightarrow \exists W\,(\mathcal{W}(W) \wedge x \in W))$.

(2) For every W with $\mathcal{W}(W)$, $\{x : \mathcal{S}(x),\ x \in W\}$ is a transitive proper subclass of $\mathcal{S}$ (for W as here we simply write "$W \in \mathcal{W}$" and identify W with $\{x : \mathcal{S}(x),\ x \in W\}$; also write "$a \in \mathcal{S}$" for $\mathcal{S}(a)$).

(3) (a) φ^W for all $W \in \mathcal{W}$, and for all (meta-mathematical quantification!) axiom φ of ZFC.

(b) For all $W \in \mathcal{W}$ and p.o. $\mathbb{P} \in W$, there is a $(W, \mathbb{P})$-generic $\mathbb{G}$ and $M[\mathbb{G}] \in \mathcal{W}$ for all such $\mathbb{G}$.

(c) For all $W \in \mathcal{W}$ if W' is a ground of W then $W' \in \mathcal{W}$ (this is formalizable by Theorem 1)

(d) $\mathcal{W}$ satisfies the amalgamation property.

The Super-$C^{(\infty)}$-LgLCAA for hyperhuge

- $\mathcal{MV}^{M,\mathbb{G}}_{\mathsf{ST}}$ depends on M and $\mathbb{G}$ and it is still a "miniature" model in that it consists of countable structures whose relation to V seems to be rather unclear.

- Assuming the Super-$C^{(\infty)}$-Laver generic Large Cardinal Axiom for All posets and for hyperhugeness (The super-$C^{(\infty)}$-LgLCAA for hyperhuge for short), we can consider a more canonical model of MV.

The super-$C^{(\infty)}$-LgLCAA for hyperhuge: For any $n \in \mathbb{N}$, for any $\lambda_0 > 2^{\aleph_0}$ and p.o. $\mathbb{P}$, there are $\lambda > \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a p.o. s.t. for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(a) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{2^{\aleph_0}} M$,
(b) $j(2^{\aleph_0}) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,
(c) (tightness) $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$, and
(d) $j''j(\lambda) \in M$ (the closure property corresponding to hyperhugeness),
(e) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and $V^{\mathsf{V}[\mathbb{H}]}_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

Theorem 5. (S.F.-Usuba [8] Lemma 4.5, Theorem 4.6 and Theorem 4.7,(4)) Consistency of the super-$C^{(\infty)}$-LgLCAA for hyperhuge can be established starting from ZFC + there is a 2-huge cardinal. □

Lemma 6. (S.F.-Usuba [8] rather hidden) The super-$C^{(\infty)}$-LgLCAA for hyperhuge implies CH. □

- $M \subseteq \mathsf{V}$ is a bedrock of V if it is a minimal ground of V. The bedrock is uniquely determined by Theorem 2 if it exists.

Theorem 7. (S.F.-Usuba [8], Theorem 5.2, Theorem 5.8) Assume that the super-$C^{(\infty)}$-LgLCAA for hyperhuge holds. Then bedrock $\overline{\mathsf{W}}$ exists and $(2^{\aleph_0})^{\mathsf{V}}$ is a super-$C^{(\infty)}$ hyperhuge cardinal in $\overline{\mathsf{W}}$. □

Theorem 8. (S.F.-Usuba [8], Lemma 4.1) For a model M of ZFC if κ is a super-$C^{(\infty)}$ hyperhuge cardinal in M then we have ${V_\kappa}^M \prec M$. □

[8] S.F. and Toshimichi Usuba, On Recurrence Axioms, Annals of Pure and Applied Logic, Vol.176, (10), (2025).

The super-$C^{(\infty)}$-LgLCAA for hyperhuge (3/3)

- Prior to Theorem 7 , Usuba had proved that a very large cardinal implies the existence of the bedrock.

Theorem 7a. (Usuba [4])Assume that there is a hyperhuge cardinal. Then bedrock $\overline{W}$ exists.

- Actually Theorem 7 in more general form generalizes this Theorem 7a.
- Usuba then improved Theorem 7a to:

Theorem 7b. (Usuba[9]) Assume that there is an extendible cardinal. Then bedrock $\overline{W}$ exists.

- **Problem:** Is an improvement of Theorem 7 possible which is similar to the one from Theorem 7a to Theorem 7b?

[9] Toshimichi Usuba, Extendible cardinals and the mantle, Archive for Mathematical Logic, Vol.58, (2019), 71-75.

Maximality Principle as Recurrence

- The Maximality Principle introduced by Hamkins for all p.o.s and parameters from $\mathcal{H}(2^{\aleph_0})$ (denoted as MP(all p.o.s, $\mathcal{H}(2^{\aleph_0})$)) can be characterized as the Recurrence Axiom (RcA(all p.o.s, $\mathcal{H}(2^{\aleph_0})$)):

RcA(all p.o.s, $\mathcal{H}(2^{\aleph_0})$): For any formula $\varphi = \varphi(\overline{x})$ in $\mathcal{L}_\varepsilon$, any p.o. $\mathbb{P}$, and any $\overline{a} \in \mathcal{H}(2^{\aleph_0})$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" then there is a ground W (of V) s.t. $\overline{a} \in$ W and W $\models \varphi(\overline{a})$.

Lemma 9. (S.F.-Usuba [8], Theorem 3.3,(5)) MP(all p.o.s, $\mathcal{H}(2^{\aleph_0})$) (i.e. RcA(all p.o.s, $\mathcal{H}(2^{\aleph_0})$)) implies CH.

Theorem 10. (S.F.-Usuba [8], Theorem 4.10) The super-$C^{(\infty)}$-LgLCAA for hyperhuge implies RcA(all p.o.s, $\mathcal{H}(2^{\aleph_0})$).

- Lemma 6 follows from Lemma 9 and Theorem 10. A direct proof is given in S.F. [10], Lemma 8.1).

[10] S.F., Extendible cardinals, and Laver-generic large cardinal axioms for extendibility, preprint.

Maximality Principle as Recurrence (2/2)

- Bedrock Axiom (BA) is the assertion that the bedrock exists. Under BA, we denote with $\overline{W}$ the bedrock.

Proposition 11. Suppose that MP(all p.o.s, $\mathcal{H}(2^{\aleph_0})$) and BA hold. Let $\kappa = (2^{\aleph_0})^{\mathsf{V}}$.

(1) For any $\mathbb{P} \in \overline{\mathsf{W}}$ with $|\mathbb{P}| < \kappa$ there is a $(\overline{W}, \mathbb{P})$-generic $\mathbb{G}$ in V.

(2) For any $\mathbb{P}_i \in \overline{\mathsf{W}}$ with $|\mathbb{P}_i| < \kappa$ and $(\overline{W}, \mathbb{P}_i)$-generic $\mathbb{G}_i \in \mathsf{V}$ for $i \in 2$, there is $\mathbb{P} \in \overline{\mathsf{W}}$ with $|\mathbb{P}| < \kappa$ and $(\overline{W}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ s.t. $\mathbb{G}_i \in \overline{\mathsf{W}}[\mathbb{G}]$ for $i \in 2$.

Proof. (1): W.l.o.g., the underlying set of $\mathbb{P}$ is in $\mathcal{P}_\kappa(\kappa)^{\overline{\mathsf{W}}} \subseteq \mathcal{H}(\kappa)^{\mathsf{V}}$. Since $\Vdash_{\mathbb{P}}$ "there is a $(\overline{\mathsf{W}}, \mathbb{P})$-generic filter". There is some ground W of V s.t. there is a $(\overline{\mathsf{W}}, \mathbb{P})$-generic filter in W.

(2): Let $\mathbb{P}^* \in \overline{\mathsf{W}}$ and $\mathbb{G}^*$ be s.t. $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}^*]$. Let $\mathbb{Q}$ be a p.o. in V s.t. $\Vdash_{\mathbb{Q}}$ "$|\mathbb{P}^*| < 2^{\aleph_0}$". Then we have

$$\overline{\mathsf{V}} \models \Vdash_{\mathbb{Q}} \text{“} \exists\, \underline{\mathbb{P}} \in \overline{\mathsf{W}},\ |\underline{\mathbb{P}}| < 2^{\aleph_0}, \exists\, (\overline{\mathsf{W}}, \underline{\mathbb{P}})\text{-generic } \underline{\mathbb{G}} \text{ s.t. } \mathbb{G}_i \in \overline{\mathsf{W}}[\underline{\mathbb{G}}] \text{ for } i \in 2 \text{”}.$$

By the Maximality Principle, the same statement holds in a ground of V and hence also in V. □ (Proposition 11)

The Multiverse reflected down to the geology

- Assume the super-$C^{(\infty)}$-LgLCAA for hyperhuge. Let $\kappa := (2^{\aleph_0})^{\mathsf{V}}$ and

$$\mathcal{MV} := \{ {V_\kappa}^{\overline{\mathsf{W}}}[\mathbb{G}] : \mathbb{G} \text{ is a } (\overline{\mathsf{W}}, \mathbb{P})\text{-generic filter } \in \mathsf{V} \text{ for some } \mathbb{P} \in {V_\kappa}^{\overline{\mathsf{W}}} \}.$$

  - By Theorem 7 BA holds. and $\kappa := (2^{\aleph_0})^{\mathsf{V}}$ is super-$C^{(\infty)}$ hyperhuge in $\overline{\mathsf{W}}$.
  - By Theorem 8, we have ${V_\kappa}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$.
  - By Theorem 10, $\mathsf{RcA}(\text{all p.o.s}, \mathcal{H}(2^{\aleph_0}))$ holds.
- Thus, by Proposition 11, $\mathcal{MV}$ can be recast into a model of MV (which we shall also call $\mathcal{MV}$).
- **Problem:** What can be said about $\mathcal{MV}$ beyond MV?
- We may construct $\mathcal{MV}$ in the framework of "Laver generic Maximum (LgM)", e.g.:

$$\mathsf{ZFC} + \mathsf{MP}(\text{all p.o.s}, \text{the first huperhuge cardinal in } \overline{W}) + \diamondsuit^{++,\textit{semiproper p.o.s}}_{\textit{Laver},2^{\aleph_0}}$$

with *the LgM feature* in my talk on Tuesday.

Some more quotations

> *„Nach alledem scheint also überhaupt keine kategorische Axiomatisierung der Mengenlehre zu existieren; denn die Schwierigkeit mit dem Beschränktheitsaxiom und den "höheren" Systemen wird wohl keine Axiomatik vermeiden können. Und da es kein Axiomensystem für Mathematik, Geometrie, usw. gibt, das nicht die Mengenlehre voraussetzte, so wird es wohl überhaupt keine kategorisch axiomatisierten unendlichen Systeme geben. Dieser Umstand scheint mir ein Argument für den Intuitionismus zu sein.“*

—— J. von Neumann; Eine Axiomatisierung der Mengenlehre (1925)

Note: Gödel's discussion with von Neumann on the First Incompleteness Theorem took place in September 1930.

[Translation by S.F.] "After all this, it seems that there is no categorical axiomatization of set theory at all, because no axiom system would be able to avoid the difficulty with the axiom of limitation and the "higher" objects. And since there is no axiomatic system for mathematics, geometry, etc. that does not presuppose set theory, there will probably be no categorically axiomatized infinite systems at all. This circumstance seems to me to be an argument in favor of intuitionism."

Some more quotations (2/4)

> *"Usuba's theorem (added by S.F.: Theorem 7b) is certainly evidence that there is a core, but there is some reason to be hesitant. First, the large cardinal hypothesis is "global", that is, Σ_3 rather than Σ_2, and that is essential. Second, strong evidence that there is a core should be evidence that there is a core with well-determined properties. The fact that the existence of extendible cardinals decides very little about the theory of the core weakens the evidence provided by Usuba's proof."*

—— J. Steel [7] (2024).

Some more quotations (3/4)

> *"However, there seems to be no guarantee that all of these natural properties of models converge to one single model of set theory."*

—— S.F.[11]

[11] S.F., The Set-theoretic multiverse as a mathematical plenitudinous Platonism viewpoint, Annals of the Japan Association for Philosophy of Science Vol.20, No. (2012) 1–5.

Some more quotations (4/4)

"$V = $ Ultimate-L"

"But there are still generic extensions."

(answering my question[12] if $V = $ *Ultimate-L* means that we have to give up all these nice axioms like MM^{++}.)

—— Hugh Woodin

[12] in a conversation on the way from 神泉 (Shinsen) to 東京大学駒場キャンパス (Komaba Campus of the University of Tokyo) on October 25, 2025.

Thank you for your attention!

ご清聴ありがとうございました.

Vielen Dank für die Aufmerksamkeit.