A Löwenheim-Skolem Theorem for a variation of stationary logic which implies that the continuum is very large

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The results in the following slides ...

are going to appear in a joint paper with André Ottenbereit Maschio Rodriques and Hiroshi Sakai:

[1] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, submitted.
http://fuchino.ddo.jp/papers/SDLS-x.pdf

[2] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, pre-preprint. http://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[3] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, III — more on generic large cardinals and some applications, in preparation.

The size of the continuum

- The size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large!

The size of the continuum (1/2)

- The size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large!

- provided that a "reasonable" and sufficiently strong reflection principle should hold.

Theorem 1. $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* CH. Proof

Actually $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ is equivalent with Sean Cox's Diagonal Reflection Principle for internal clubness + CH.

Theorem 2. **(a)** $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0}=\aleph_2$. Proof

(b) $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to Diagonal Reflection Principle for internal clubness* **(c)** $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *is equivalent to* $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)+\neg\mathrm{CH}$. Proof

Theorem 3. $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0}$ *is very large (e.g. weakly Mahlo, weakly hyper Mahlo, etc.)* Proof

The size of the continuum (2/2)

- The size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large!

  - provided that a strong variant of generic large cardinal exists.

Theorem 1. *If there exists a Laver-generically supercompact cardinal κ for σ-closed p.o.s, then $\kappa = \aleph_2$ and* CH *holds. Moreover $MA^{+\aleph_1}(\sigma\text{-closed})$ holds. Thus $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ follows.*

Theorem 2. *If there exists a Laver-generically supercompact cardinal κ for proper p.o.s, then $\kappa = \aleph_2 = 2^{\aleph_0}$. Moreover $\mathsf{PFA}^{+\aleph_1}$ holds. Thus $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ follows.*

Theorem 3. *If there exists a Laver generically supercompact cardinal κ for c.c.c. p.o.s, then $\kappa \leq 2^{\aleph_0}$ and κ is very large (e.g. weakly Mahlo, weakly hyper Mahlo, etc.) Moreover $\mathsf{MA}^{+\mu}(ccc, <\kappa)$ for all $\mu < \kappa$ and $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ hold.*

Theorem 1. **(1)** *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for σ-closed p.o.s" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for proper p.o.s" is consistent as well.*

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a strongly Laver-generically supercompact cardinal for c.c.c. p.o.s" is consistent as well.*

Proof. Starting from a model of ZFC with a supercompact cardinal κ (a superhuge cardinal in case of (2)), we can obtain models of respective assertions by iterating (in countable support in case of (1), (2) and in finite support in case of (3)) with respective p.o.s κ times along a Laver function (for (1) and (2) Laver function for supercompactness; for (2), Laver function for super-almost-hugeness). □

Conclusion

- The size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large

  - provided that a "reasonable" and sufficiently strong reflection principle should hold.

  - These “reasonable” reflection principle in terms of Löwenheim Skolem Theorem are consequences of respective axioms of generic supercompactness.

Some more background and open problems

- By a slight modification of B. König's results, the implication of SDLS($\mathcal{L}^{\aleph_0}_{stat}, \leq \aleph_2$) from the existence of Laver-generically supercompact cardinal can be interpolated by a Game Reflection Principle which by itself characterizes the usual version of generically supercompactness of $\aleph_2$.
- ▷ Do there exist some Game Reflection Principles which play similar role in the other two scenerios in the trichotomy?
- ▷ Does (Strong) Laver-generially supercompactness of κ for c.c.c. p.o.s imply $\kappa = 2^{\aleph_0}$?

Thank you for your attention.

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$\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$ implies $2^{\aleph_0}$ is very large.

- For a regular cardinal κ and a cardinal $\lambda \geq \kappa$, $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$ is said to be 2-*stationary* if, for any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there is an $a \in \mathcal{S}$ s.t. $|\kappa \cap a|$ is a regular uncountable cardinal and $\mathcal{T} \cap \mathcal{P}_{\kappa\cap a}(a)$ is stationary in $\mathcal{P}_{\kappa\cap a}(a)$. A regular cardinal κ has the 2-*stationarity property* if $\mathcal{P}_\kappa(\lambda)$ is 2-stationary (as a subset of itself) for all $\lambda \geq \kappa$.

Lemma 1. *For a regular uncountable κ, $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ implies that κ is 2-stationary.*

Lemma 2. *Suppose that κ is a regular uncountable cardinal.*
(1) *If κ is 2-stationary then κ is a limit cardinal.*
(2) *For any $\lambda \geq \kappa$, 2-stationary $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$, and any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there are stationarily many $r \in \mathcal{S}$ s.t. $\mathcal{T} \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.*
(3) *If κ is 2-stationary then κ is a weakly Mahlo cardinal.*

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$\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ is equivalent to $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) + \neg\mathrm{CH}$.

- If $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds then $2^{\aleph_0} = \aleph_2$ by (a). Thus, it follows that $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2) + \neg\mathrm{CH}$ holds.
- Suppose $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ holds. Then we have $2^{\aleph_0} \leq \aleph_2$ by a theorem of Todorčević already mentioned. Thus, if $2^{\aleph_0} > \aleph_1$ in addition, we have $2^{\aleph_0} = \aleph_2$. Thus $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ follows.

□

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Baumgartner's Theorem

Theorem 1 (J.E. Baumgartner). *Let $\omega < \kappa < \lambda$ and κ be regular. Then any club subset of $[\lambda]^{<\kappa}$ has cardinality $\geq \lambda^{\aleph_0}$.*

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$\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ for $\kappa > \aleph_2$ implies $\kappa > 2^{\aleph_0}$.

- $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies $2^{\aleph_0} \leq \aleph_2$: it is easy to see that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the reflection principle $\mathrm{RP}(\omega_2)$ in [millennium-book]. $\mathrm{RP}(\omega_2)$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević). Thus we have $\kappa > \aleph_2 \geq 2^{\aleph_0}$.
- Thus we may assume that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ does not hold. Hence there is a structure $\mathfrak{A}$ s.t., for any $\mathfrak{B} \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}$, we have $\|\mathfrak{B}\| \geq \aleph_2$. Let $\lambda = \|\mathfrak{A}\|$. W.l.o.g., we may assume that the underlying set of $\mathfrak{A}$ is $= \lambda$. Let $\mathfrak{A}^* = \langle \mathcal{H}(\lambda^+), \underbrace{\lambda, ..., \in}_{=\mathfrak{A}} \rangle$.
- By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$, there is $M \in [\mathcal{H}(\lambda^+)]^{<\kappa}$ s.t. $\mathfrak{A}^* \restriction M \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}^*$. It follows that $\mathfrak{A} \restriction (\lambda \cap M) \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}$. By the choice of $\mathfrak{A}$, we have $|M| \geq |\lambda \cap M| \geq \aleph_2$.
- By elementarity, there is $C \subseteq [M]^{\aleph_0} \cap M$ which is a club in $[M]^{\aleph_0}$. By a theorem of Baumgartner, it follows that $\kappa > |M| \geq |C| \geq 2^{\aleph_0}$.

□

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$\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ implies $2^{\aleph_0} = \aleph_2$.

Proposition 1. $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ *for* $\kappa > \aleph_2$ *implies* $\kappa > 2^{\aleph_0}$.

Proof

- Suppose that $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds. Then $2^{\aleph_0} \leq \aleph_2$ by the Proposition 1.
- $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_1)$ does not hold since

  "there are uncountably many x s.t. ..."

  is expressible in $\mathcal{L}^{\aleph_0}_{stat}$. Thus, $2^{\aleph_0} > \aleph_1$.

□

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$\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ implies CH.

- Suppose that $\mathfrak{A} = \langle \mathcal{H}(\omega_1), \in \rangle$ and Let $B \in [\mathcal{H}(\omega_1)]^{<\aleph_2}$ be s.t. $\mathfrak{A} \restriction B \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. Then for any $U \in [B]^{\aleph_0}$ we have $\mathfrak{A} \models$ "$\exists x \, \forall y \, (y \in x \leftrightarrow y \, \varepsilon \, U)$".
- By elementarity we also have $\mathfrak{B} \models$ "$\exists x \, \forall y \, (y \in x \leftrightarrow y \, \varepsilon \, U)$".
- ▷ It follows that $U \in B$. Thus $[B]^{\aleph_0} \subseteq B$ and $2^{\aleph_0} \leq |\, B \,| \leq \aleph_1$. □

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Strong Downward Löwneheim-Skolem Theorem for stationary logic

- $\mathcal{L}^{\aleph_0}_{stat}$ is a weak second order logic with monadic second-order variables X, Y etc. which run over the countable subsets of the underlying set of a structure. The logic has only the weak second order quantifier "*stat*" and its dual "*aa*" (but not the second-order existential (or universal) quantifiers) with the interpretation:

  $\mathfrak{A} \models stat\, X\, \varphi(..., X) \quad :\Leftrightarrow$
  $\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(..., U)\}$ is a stationary subset of $[A]^{\aleph_0}$.

- For $\mathfrak{B} = \langle B, ... \rangle \subseteq \mathfrak{A}$, $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models \varphi(a_0, ..., U_0, ...) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, ..., U_0, ...)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$ and for all $a_0, ... \in B$ and for all $U_0, ... \in [B]^{\aleph_0}$.

- SDLS$(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \quad :\Leftrightarrow$
  For any structure $\mathfrak{A} = \langle A, ... \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

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A weakening of the Strong Downward Löwneheim-Skolem Theorem

- For $\mathfrak{B} = \langle B, ... \rangle \subseteq A$, $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$

  $\mathfrak{B} \models \varphi(a_0, ...) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, ...)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ...)$ without free seond-order variables and for all $a_0, ... \in B$.

- $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \quad :\Leftrightarrow$

  For any structure $\mathfrak{A} = \langle A, ... \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

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Strong Downward Löwneheim-Skolem Theorem for PKL logic

- $\mathcal{L}^{PKL}_{stat}$ is the weak second-order logic with monadic second order variables X, Y, etc. with built-in unary predicate symbol $\underset{\sim}{K}$. The monadic seond order variables run over elements of $\mathcal{P}_{\underset{\sim}{K}^{\mathfrak{A}}}(A)$ for a structure $\mathfrak{A} = \langle A, \underset{\sim}{K}^{\mathfrak{A}}, ... \rangle$ where we denote $\mathcal{P}_S(T) = \mathcal{P}_{|S|}(T) = \{u \subseteq T : |u| < |S|\}$. The logic has the unique second order quantifier "*stat*" (and its dual).

- For $\mathfrak{B} = \langle B, K \cap B, ... \rangle \subseteq \mathfrak{A} = \langle A, K, ... \rangle$, $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$ $\mathfrak{B} \models^{int} \varphi(a_0, ..., U_0, ...) \Leftrightarrow \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ...)$ $a_0, ... \in B$ and $U_0, ... \in \mathcal{P}_{K \cap B}(B) \cap B$.

- $\mathrm{SDLS}^{int}(\mathcal{L}^{PKL}_{stat}, < \kappa) \quad :\Leftrightarrow$ for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, ... \rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$. $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

Strong Downward Löwneheim-Skolem Theorem for PKL logic (2/2)

- $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa) \quad :\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, ... \rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$. $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$, there are stationarily many structures $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

- ▷ The internal interpretation of the quantifier is defined by:

  $\mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, ..., U_0, ..., X) \quad :\Leftrightarrow$
  $\{U \in \mathcal{P}_{\underline{K}}\mathfrak{A}(A) \cap A : \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\}$ is a stationary subset of $\mathcal{P}_{\underline{K}}\mathfrak{A}(A)$ for $a_0, ...A$ and $U_0, ... \in \mathcal{P}_{\underline{K}}\mathfrak{A}(A) \cap A$.

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Laver generically supercompact cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a Laver-generically supercomact for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{Q} = \mathbb{P} * \underset{\sim}{\mathbb{R}}$, s.t., for any $(V, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq V[\mathbb{H}]$, and an elementary embedding $j : V \to M$ s.t.

  (1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

  (2) $\mathbb{P}, \mathbb{H} \in M$,

  (3) $j''\lambda \in M$.

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Diagonal Reflection Principle

- For a regular cardinal $\theta > \aleph_1$:

  DRP(θ, IC): There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ s.t.

  (1) $M \cap \mathcal{H}(\theta)$ is internally club;

  (2) for all $R \in M$ s.t. R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

- For a regular cardinal $\lambda > \aleph_1$

  $(*)_\lambda$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$, if $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$, is a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$, then there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, for all $a \in M$.

Proposition 1. *TFAE:* (a) *The global version of Diagonal Reflection Principle of S.Cox for internal clubness (i.e.* DRP(θ, IC) *for all regular* $\theta > \aleph_1$*) holds.*

(b) $(*)_\lambda$ *for all regular* $\lambda > \aleph_1$ *holds.*

(c) $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *holds.*

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Reflection Principles $\mathrm{RP}_{??}$

- The following are variations of the "Reflection Principle" in [Jech, Millennium Book].

$\mathrm{RP}_{\mathrm{IC}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

$\mathrm{RP}_{\mathrm{IU}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally unbounded $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

Since every internally club M is internally unbounded, we have:

Lemma 1. $\mathrm{RP}_{\mathrm{IC}}$ *implies* $\mathrm{RP}_{\mathrm{IU}}$.

$\mathrm{RP}_{\mathrm{IU}}$ is also called Axiom R in Set-Theoretic Topology.

Theorem 2. ([?]) $\mathrm{RP}_{\mathrm{IU}}$ *implies* FRP.

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Stationary subsets of $[X]^{\aleph_0}$

- $C \subseteq [X]^{\aleph_0}$ is club in $[X]^{\aleph_0}$ if (1) for every $u \in [X]^{\aleph_0}$, there is $v \in C$ with $u \subseteq v$; and (2) for any countable increasing chain $\mathcal{F}$ in C we have $\bigcup \mathcal{F} \in C$.

- $S \subseteq [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$ if $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^{\aleph_0}$.

- A set M is internally unbounded if $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$ (w.r.t. $\subseteq$)

- A set M is internally stationary if $M \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$

- A set M is internally club if $M \cap [M]^{\aleph_0}$ contains a club in $[M]^{\aleph_0}$.

"Diagonal Reflection Principle" にもどる

"RP_{η}" にもどる

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\kappa_\omega$ and any mapping $g : E \to [\kappa]^{\aleph_0}$ with $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$, there is $\gamma \in E^\kappa_{\omega_1}$ s.t.

(*) for any $I \in [\gamma]^{\aleph_1}$ closed w.r.t. g and club in γ, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$.

- ▷ $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |I|$ s.t. $I = \bigcup_{\alpha<\lambda} I_\alpha$.
- ▶ FRP follows from Martin's Maximum or Rado's Conjecture. $\mathrm{MA}^+(\sigma\text{-closed})$ already implies FRP but PFA does not imply FRP since PFA does not imply stationary reflection of subsets of $E^{\omega_2}_\omega$ (Magidor, Beaudoin) which is a consequence of FRP.
- ▶ FRP is a large cardinal property: FRP implies the total failure of the square principle.
- ▷ FRP is known to be equivalent to the reflection of uncountable coloring number of graphs down to cardinality $< \aleph_2$.

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Proof of Fact 1

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal κ there is a non-metrizable space X of size κ s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $\kappa' \geq \kappa$ be of cofinality $\geq \kappa,\ \omega_1$.
- The topological space $(\kappa' + 1, \mathcal{O})$ with

$$\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} : x \subseteq \kappa',\ x \text{ is bounded in } \kappa'\}$$

  is non-metrizable since the point κ' has character $= \mathrm{cf}(\kappa') > \aleph_0$.
- Any subspace of $\kappa' + 1$ of size $< \kappa$ is discrete and hence metrizable.

□

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Proof of Fact 3

- It is enough to prove the following:

Lemma 1. (Folklore ?, see [?]) *For a regular cardinal $\kappa \geq \aleph_2$ if, there is a non-reflectingly stationary $S \subseteq E^\kappa_\omega$, then there is a non meta-lindelöf (and hence non metrizable) locally compact and locally countable topological space X of cardinality κ s.t. all subspace Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $I = \{\alpha + 1 : \alpha < \kappa\}$ and $X = S \cup I$.
- ▷ Let $\langle a_\alpha : \alpha \in S \rangle$ be s.t. $a_\alpha \in [I \cap \alpha]^{\aleph_0}$, a_α is of order-type ω and cofinal in α. Let $\mathcal{O}$ be the topology on X introduced by letting

  (1) elements of I are isolated; and

  (2) $\{a_\alpha \cup \{\alpha\} \setminus \beta : \beta < \alpha\}$ a neighborhood base of each $\alpha \in S$.

- Then $(X, \mathcal{O})$ is not meta-lindelöf (by Fodor's Lemma) but each $\alpha < \kappa$ as subspace of X is metrizable (by induction on α).□ もどる

Coloring number and chromatic number of a graph

- For a cardinal $\kappa \in \mathrm{Card}$, a graph $G = \langle G, K \rangle$ has coloring number $\leq \kappa$ if there is a well-ordering $\sqsubseteq$ on G s.t. for all $p \in G$ the set

$$\{q \in G : q \sqsubseteq p \text{ and } q\, K\, p\}$$

  has cardinality $< \kappa$.

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- ▷ The coloring number $col(G)$ of a graph G is the minimal cardinal among such κ as above.
- The chromatic number $chr(G)$ of a graph $G = \langle G, K \rangle$ is the minimal cardinal κ s.t. G can be partitioned into κ pieces $G = \bigcup_{\alpha<\kappa} G_\alpha$ s.t. each G_α is pairwise non adjacent (independent).
- ▷ For all graph G we have $chr(G) \leq col(G)$.

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κ-special trees

- For a cardinal κ, a tree T is said to be κ-special if T can be represented as a union of κ subsets T_α, $\alpha < \kappa$ s.t. each T_α is an antichain (i.e. pairwise incomparable set).

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Stationary subset of E^κ_ω

- For a cardinal κ,

$$E^\kappa_\omega = \{\gamma < \kappa : \mathrm{cf}(\gamma) = \omega\}.$$

- A subset $C \subseteq \xi$ of an ordinal ξ of uncountable cofinality, C is closed unbounded (club) in ξ if (1): C is cofinal in ξ (w.r.t. the canonical ordering of ordinals) and (2): for all $\eta < \xi$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.

- $S \subseteq \xi$ is stationary if $S \cap C \neq \emptyset$ for all club $C \subseteq \xi$.

- A stationary $S \subseteq \xi$ if reflectingly stationary if there is some $\eta < \xi$ of uncountable cofinality s.t.$S \cap \eta$ is stationary in η. Thus:

- A stationary $S \subseteq \xi$ if non reflectingly stationary if $S \cap \eta$ is non stationary for all $\eta < \xi$ of uncountable cofinality.

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Proof of Theorem 1.

CH $\Rightarrow$ SDLS($\mathcal{L}^{\aleph_0,II}, < \aleph_2$): For a structure $\mathfrak{A}$ with a countable signature L and underlying set A, let θ be large enough and $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\theta), A, \mathfrak{A}, \in \rangle$. where $A = \underline{A}^{\tilde{\mathfrak{A}}}$ for a unary predicate symbol $\underline{A}$ and $\mathfrak{A} = \underline{\mathfrak{A}}^{\tilde{\mathfrak{A}}}$ for a constant symbol $\underline{\mathfrak{A}}$. Let $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ be s.t. $|B| = \aleph_1$ for the underlying set B of $\mathfrak{B}$ and $[B]^{\aleph_0} \subseteq B$. $\mathfrak{B} = \mathfrak{A} \restriction \underline{A}^{\tilde{\mathfrak{B}}}$ is then as desired.

SDLS($\mathcal{L}^{\aleph_0}, < \aleph_2$) $\Rightarrow$ CH: Suppose $\mathfrak{A} = \{\omega_2 \cup [\omega_2]^{\aleph_0}, \in\}$. Consider the $\mathcal{L}^{\aleph_0}$-formula $\varphi(X) = \exists x \forall y \, (y \in x \leftrightarrow y \, \varepsilon \, X)$.
If $\mathfrak{B} = \langle B, \ldots \rangle$ is s.t. $|B| \leq \aleph_1$ and $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}}$, then for $C \in [B]^{\aleph_0}$, since $\mathfrak{A} \models \varphi(C)$, we have $\mathfrak{B} \models \varphi(C)$. It dollows that $[B]^{\aleph_0} \subseteq B$ and $2^{\aleph_0} \leq (|B|)^{\aleph_0} \leq |B| = \aleph_1$.

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