Definability of Laver-generic large cardinals and largeness of generic large cardinals with chain conditions

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Outline

Generic and Laver-generic large cardinals

- For a class $\mathcal{P}$ of p.o.s, we call a cardinal κ generically supercompact by $\mathcal{P}$ ($\mathcal{P}$-gen. supercompact, for short) if, for any $\lambda \geq \kappa$, there are $\mathbb{P} \in \mathcal{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. ① $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$, ② $j(\kappa) > \lambda$, and ③ $j''\lambda \in M$.

- A cardinal κ is Laver-generically supercompact for $\mathcal{P}$ (or $\mathcal{P}$-Laver-gen. supercompact, for short) if, for any $\lambda \geq \kappa$, $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for all $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H} \supseteq \mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t.
① $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$, ② $j(\kappa) > \lambda$, and ③' $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H}$, $j''\lambda \in M$.

▷ **Notation.** We denote with $j : N \overset{\preceq}{\to}_\kappa M$ the situation that N and M are transitive (sets or classes); j is an elementary embedding of the structure $\langle N, \in \rangle$ into the structure $\langle M, \in \rangle$; $\kappa \in N$, and $crit(j) = \kappa$.

Generic and Laver-generic large cardinals (2/3)

- For a class $\mathcal{P}$ of p.o.s, we call a cardinal κ $\mathcal{P}$-gen. supercompact $\mathcal{P}$-gen. super-almost-huge $\mathcal{P}$-gen. superhuge if, for any $\lambda \geq \kappa$, there are $\mathbb{P} \in \mathcal{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③ $j''\lambda \in M$ $j''\mu \in M$ for all $\mu < j(\kappa)$ $j''j(\kappa) \in M$.

- A cardinal κ is $\mathcal{P}$-Laver-gen. supercompact $\mathcal{P}$-Laver-gen. supe-almost-huge $\mathcal{P}$-Laver-gen. superhuge if, for any $\lambda \geq \kappa$, $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there is a $\mathcal{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for all $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H} \supseteq \mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③' $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$ and, $j''\lambda \in M$ $j''\mu \in M$ for all $\mu < j(\kappa)$ $j''j(\kappa) \in M$.

▷ $\mathcal{P}$-Laver-gen. superhuge $\Rightarrow$ $\mathcal{P}$-Laver-gen. super-almost-huge $\Rightarrow$ $\mathcal{P}$-Laver-gen. supercompact

$\Downarrow$ $\Downarrow$ $\Downarrow$

$\mathcal{P}$-gen. superhuge $\Rightarrow$ $\mathcal{P}$-gen. super-almost-huge $\Rightarrow$ $\mathcal{P}$-gen. supercompact

Generic and Laver-generic large cardinals (3/3)

Notation:

- For $\mathcal{P} = \{\mathbb{P} : \mathbb{P}$ is σ-closed$\}$, we say σ-closed-gen. supercompact, or σ-closed-Laver-gen. supercompact, etc. in place of $\mathcal{P}$-gen. supercompact, or $\mathcal{P}$-Laver-gen. supercompact, etc.

  - For $\mathcal{P} = \{\mathbb{P} : \mathbb{P}$ is μ-cc$\}$, we say μ-cc-Laver-gen. supercompact, etc. instead of $\mathcal{P}$-Laver-gen. supercompact, etc.

  - Similarly, we say Cohen-Lever-gen. supercompact etc. instead of $\mathcal{P}$-Laver-gen. supercompact for $\mathcal{P} = \{\mathrm{Fn}(\lambda, 2) : \lambda \in \mathsf{On}\}$.

- For a p.o. $\mathbb{P}$, we say $\mathbb{P}$-gen. supercompact etc. instead of $\{\mathbb{P}\}$-gen. supercompact etc.

Some standard models of Laver-generic large cardinals

a) Suppose κ is supercompact and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is σ-closed-Laver-gen. supercompact and CH holds (similarly for super-almost-huge, or superhuge).

b) Suppose κ is super-almost-huge with a Laver function f, and $\mathbb{P}$ is the CS-iteration for forcing PFA along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is proper-Laver-generically super-almosthuge and $2^{\aleph_0} = \aleph_2$ holds (similarly for superhuge).

c) Suppose κ is supercompact and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $(2^{\aleph_0})^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is Cohen-Laver-generically supercompact (similarly for super-almost-huge, or superhuge). $\kappa = 2^{\aleph_0}$ is very large

d) Suppose that κ is supercompact with a Laver function f, and $\mathbb{P}$ is a FS-iteration for forcing MA along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is ccc-Laver-generically supercompact (similarly for super-almost-huge, or superhuge). $\kappa = 2^{\aleph_0}$ is very large

- See [II] for more details. tightness

The Trichotomy Theorem

- The examples in the previous slide can be considered as instances of the following more general situations.

Theorem 1. (A) If κ is $\mathcal{P}$-Laver-gen. supercompact for a class $\mathcal{P}$ of p.o.s such that ① all $\mathbb{P} \in \mathcal{P}$ are ω_1 preserving, ② all $\mathbb{P} \in \mathcal{P}$ do not add reals, and ③ there is a $\mathbb{P}_1 \in \mathcal{P}$ which collapses ω_2, then $\kappa = \aleph_2$ and CH holds.

(B) If κ is $\mathcal{P}$-Laver-gen. supercompact for a class $\mathcal{P}$ of p.o.s such that ① all $\mathbb{P} \in \mathcal{P}$ are ω_1-preserving, ②' there is a $\mathbb{P}_0 \in \mathcal{P}$ which add a real, and ③ there is a $\mathbb{P}_1$ which collapses ω_2, then $\kappa = \aleph_2 = 2^{\aleph_0}$.

(C) If κ is $\mathcal{P}$-Laver-gen. supercompact for a class $\mathcal{P}$ of p.o.s such that ①' all $\mathbb{P} \in \mathcal{P}$ preserve cardinals, and ②' there is a $\mathbb{P}_0 \in \mathcal{P}$ which adds a real, then κ is "very large" and $\kappa \leq 2^{\aleph_0}$.

Proof. The proof is contained implicitly in [II]. For a more explicit presentation of a proof, see [Nagoya]. □

Tight Laver-genericity

- (C) of Theorem 1 on the previous slide can be improved for a (slightly?) stronger variation of Laver-genericity:

- A cardinal κ is tightly $\mathcal{P}$-Laver-gen. superhuge
  if, for any $\lambda \geq \kappa$, $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there is a $\mathcal{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for all $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H} \supseteq \mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. ① $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$,
  ②' $|\, \mathbb{P} * \underset{\sim}{\mathbb{Q}}\, | \leq j(\kappa)$, $j(\kappa) > \lambda$, and ③' $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j''j(\kappa) \in M$.

- All the examples a)∼d) of Laver-genericity give actually tightly Laver-generic cardinals.

Theorem 2. ([II]) Suppose that each element of $\mathcal{P}$ is μ-cc for some $\mu < \kappa$. If κ is $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$.

Proof.

Definability of Laver-generic large cardinals

- The statement "there is a $\mathcal{P}$-Laver gen. supercompact (super-almost huge, or superhuge) cardinal" is expressible as a first-order statement in the framework of ZFC.

- ▷ This follows from the Proposition 3 and Lemma 4, 5 below:

- The following Proposition 3 seems to be a folklore. It can be proved using a "generic" variant of the idea of extender:

Proposition 3. ([Def]) Suppose that $\mathbb{P}$ is a p.o. (in V) and $\mathbb{G}$ a $(\mathsf{V},\mathbb{P})$-generic filter. Suppose further that θ is a regular cardinal and $j_0 : \mathcal{H}(\theta)^{\mathsf{V}} \overset{\preceq}{\to} N$ for a transitive set N with j_0, $N \in \mathsf{V}[\mathbb{G}]$ is s.t. ① $\mathbb{P} \in \mathcal{H}(\theta)^{\mathsf{V}}$; and
② for any $b \in N$, there is $a \in \mathcal{H}(\theta)^{\mathsf{V}}$ s.t. $b \in j_0(a)$.

Then there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. ③ $j : \mathsf{V} \overset{\preceq}{\to} M$, ④ $N \subseteq M$, and ⑤ $j \restriction \mathcal{H}(\theta)^{\mathsf{V}} = j_0$.

A sketch of the Proof

Definability of Laver-generic large cardinals (2/3)

- The following Lemmata are easy to prove:

Lemma 4. ([Def]) Suppose that $\mathbb{P}$ is a p.o. (in V), and $\mathbb{G}$ a $(\mathsf{V},\mathbb{P})$-generic set. Suppose that $j, M \subseteq \mathsf{V}[\mathbb{G}]$ are s.t. $j : \mathsf{V} \overset{\prec}{\to} M$
Then, for any cardinal θ (in V), we have:

$$\mathsf{V}[\mathbb{G}] \models \text{“} j \restriction \mathcal{H}(\theta)^{\mathsf{V}} : \mathcal{H}(\theta)^{\mathsf{V}} \overset{\prec}{\to} \mathcal{H}(j(\theta))^M \text{”}.$$

- The extra condition ② in Proposition 3 can be handled by the following:

Lemma 5. ([Def]) Suppose that $\mathbb{P}$ is a p.o. (in V), and $\mathbb{G}$ a $(\mathsf{V},\mathbb{P})$-generic set. Suppose further that θ is a regular cardinal in V and j_0, $N \in \mathsf{V}[\mathbb{G}]$ be such that N is transitive and $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N$.
Let $N_0 = \bigcup j_0{}''\mathcal{H}(\theta)^{\mathsf{V}}$. Then, we have:
① N_0 is transitive. ② (i) $N_0 \prec N$, (ii) $j_0{}''\mathcal{H}(\theta) \subseteq N_0$, and (iii) $j_0 : \mathcal{H}(\theta)^{\mathsf{V}} \overset{\prec}{\to} N_0$. ③ For any $b \in N_0$ there is $a \in \mathcal{H}(\theta)^{\mathsf{V}}$ s.t. $b \in j_0(a)$.
④ If $\theta_0 < \theta$ is s.t. $\mathcal{H}(\theta_0)^{\mathsf{V}} \in \mathcal{H}(\theta)^{\mathsf{V}}$, then $(\mathcal{H}(j_0(\theta_0)))^N \subseteq N_0$.

Definability of Laver-generic large cardinals (3/3)

- Putting together Proposition 3, Lemma 4 and Lemma 5, we obtain the following:

Theorem 6. ([Def]) For a class $\mathcal{P}$ of p.o.s, the following are equivalent:

(a) κ is $\mathcal{P}$-Laver-gen. supercompact;

(b) For any $\lambda \geq \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t.

$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "there are a regular cardinal $\theta > \kappa$, a transitive set N, and a mapping j_0 s.t.

① $j_0 : \mathcal{H}(\theta)^{\mathsf{V}} \xrightarrow{\preccurlyeq} N$,

② $crit(j_0) = \kappa$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{H}(\theta)$, $j_0(\kappa) > \lambda$,

③ for any $b \,\varepsilon\, N$, there is $a \,\varepsilon\, \mathcal{H}(\theta)^{\mathsf{V}}$ s.t. $b \,\varepsilon\, j_0(a)$

④ $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\underset{\sim}{\mathbb{H}} \in N$, and ⑤ $j_0{}''\lambda \in N$ ".

- A similar equivalence holds also for $\mathcal{P}$-Laver gen. super-almost-huge and superhuge cardinals.

Small generic large cardinals

- For a class $\mathcal{P}$ of p.o.s, we say that a cardinal κ is $\mathcal{P}$-gen. weakly compact, if, for any $A \subseteq \kappa$ ($A \in \mathsf{V}$), there is a transitive set model M of ZFC$^-$ with $\kappa, A \in M$ s.t., for some $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $j : M \overset{\preceq}{\to}_\kappa N$ for some j, $N \in V[\mathbb{G}]$. ([CC])

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is $\mathcal{P}$-gen. measurable, if there is $\mathbb{P} \in \mathcal{P}$ s.t., for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ with $\mathsf{V}[\mathbb{G}] \models$"$j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$".

▷ For any class $\mathcal{P}$ of p.o.s, we have

$$\kappa \text{ is } \mathcal{P}\text{-Laver-gen. supercompact} \;\Rightarrow\; \kappa \text{ is } \mathcal{P}\text{-gen. supercompact} \;\Rightarrow\; \kappa \text{ is } \mathcal{P}\text{-gen. measurable} \;\Rightarrow\; \kappa \text{ is } \mathcal{P}\text{-gen. weakly-compact.}$$

by definition.

- In the following we present results which show that for

(†) $\mathcal{P} \subseteq \{\mathbb{P} : \mathbb{P} \text{ satisfies } \mu\text{-cc for some } \mu < \kappa\}$

the implications above are very much "strict".

- If

($\dagger$) $\mathcal{P} \subseteq \{\mathbb{P} : \mathbb{P}$ satisfies μ-cc for some $\mu < \kappa\}$,

all $\mathcal{P}$-gen. weakly-compact cardinals are already fairly large:

Lemma 7. ([CC]) Suppose that $\mathcal{P}$ satisfies ($\dagger$) and κ is $\mathcal{P}$-gen. weakly-compact. Then (1) κ is weakly Mahlo. (2) κ has the tree property.

Proof.

Theorem 8. ([CC]) Suppose that $\mathcal{P}$ satisfies ($\dagger$) and κ is $\mathcal{P}$-gen. measurable. Then κ is a stationary limit of $\mathcal{P}$-gen. weakly-compact cardinals.

Theorem 9. ([CC]) Suppose that $\mathcal{P}$ satisfies ($\dagger$) and κ is $\mathcal{P}$-gen. supercompact. Then κ is a stationary limit of $\mathcal{P}$-gen. measurable cardinals.

- The proof of Theorem 8 uses the following characterization of $\mathcal{P}$-gen. measurable cardinals for $\mathcal{P}$ satisfying $(\dagger)$:

Theorem 10. ([CC]) For a regular cardinals κ, ν with $\nu < \kappa$, the following are equivalent:

(a) κ is ν-cc-gen. measurable.

(b) There is a non-trivial, non-principal and ν-saturated $<\kappa$-complete ideal over κ.

(c) there are ν-cc p.o. $\mathbb{P}$, $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models$ "$j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$" and $({}^{\kappa}M)^{\mathsf{V}[\mathbb{G}]} \subseteq M$.

Historical background

- There are several other authors who considered some variations of generic large cardinals as new axioms of set theory, notably Matt Foreman and Bernhard König. https://arxiv.org/abs/1403.2788
- Laver-genericity shows some reminiscence of Resurrection Axioms introduced and studied by [Hamkins and Johnstone], [Hamkins and Johnstone 2] (see also [Minden] and [Tsaprounis]). The similarity was already pointed out by Joel when I gave a talk (in person) in 2019 at the NY Set Theory Seminar.
- I first heard the idea of resurrection axioms in 2015 from Joel Hamkins when we had a long walk to and through Yamashita Park (山下公園) in Yokohama. In retrospective, this might have played subliminally an important role in me when I invented the primary version of Laver-genericity in 2018 and began to discuss it with Hiroshi and also with Andrés Ottenbreit Maschio Rodrigues, a PhD student of mine back then.

Historical background — September 12, 2015

- These Axioms are different in that Resurrection Axioms are absoluteness statements while the existence of a/the Laver-generic large cardinal is a Reflection Axiom. It seems to be still open what the connections between these two types of axioms can be ([added after the talk] we know now more about what these connections are: see the following additional slides).

Resurrection Axioms (slides added after the talk)

- For a class $\mathcal{P}$ of p.o.s and a definable cardinal μ (e.g. defined to be $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the Resurrection Axiom for $\mathcal{P}$ and $\mathcal{H}(\mu)$ is defined by:

$\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}(\mu)}$: For any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, we have $\mathcal{H}(\mu)^{\mathsf{V}} \prec \mathcal{H}(\mu)^{\mathsf{V}[\mathbb{H}]}$.

- Here, μs in the left and right side of the last formula are actually meant μ^{V} and $\mu^{\mathsf{V}[\mathbb{H}]}$ respectively.

Theorem 11. (1) For a class of p.o.s $\mathcal{P}$ satisfying the conditions in (A) of Theorem 1, if κ ($\aleph_2 = (2^{\aleph_0})^+$ see Theorem 1) is tightly $\mathcal{P}$-Laver-gen. superhuge, then $\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}((2^{\aleph_0})^+)}$ holds.

(2) For a class of p.o.s $\mathcal{P}$ satisfying the conditions in one of (B) or (C) of Theorem 1, if κ ($= 2^{\aleph_0}$ see Theorem 1 and 2) is tightly $\mathcal{P}$-Laver-gen. superhuge, then $\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}(2^{\aleph_0})}$ holds.

Proof. (The following proof is based on the idea suggested by Gunter Fuchs during the talk).

- (1) and (2) are proved similarly. We give here a proof of (1).
- Suppose $\mathbb{P} \in \mathcal{P}$. Then, by tightly $\mathcal{P}$-Laver gen. superhugeness of κ ($= (2^{\aleph_0})^+$), there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. with $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with ① $j : V \overset{\prec}{\rightarrow}_\kappa M$, ② $j(\kappa) = |\mathbb{P} * \underset{\sim}{\mathbb{Q}}|$, ③ $\mathbb{P}$, $\mathbb{H} \in M$ and ④ $j''j(\kappa) \in M$.

Claim. $\mathcal{H}(j(\kappa))^{\mathsf{V}[\mathbb{H}]} \subseteq M$ and hence $\mathcal{H}(j(\kappa))^M = \mathcal{H}(j(\kappa))^{\mathsf{V}[\mathbb{H}]}$.

$\vdash$ Suppose that $b \in \mathcal{H}(j(\kappa))^{\mathsf{V}[\mathbb{H}]}$ and let $c \subseteq j(\kappa)$ be a code of b. Let $\underset{\sim}{c}$ be a nice $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-name of c. By ②, $|\underset{\sim}{c}| \leq j(\kappa)$. By ④ it follows that $\underset{\sim}{c} \in M$. Thus $c \in M$ by ③ , and hence $b \in M$. $\dashv$

- Since $crit(j) = \kappa$, $j(a) = a$ for all $a \in (\mathcal{H}(\kappa))^{\mathsf{V}}$. Thus $id_{\mathcal{H}(\kappa)} = j \restriction \mathcal{H}(\kappa)^{\mathsf{V}} : \mathcal{H}(\kappa)^{\mathsf{V}} \overset{\prec}{\rightarrow} \mathcal{H}(j(\kappa))^{\mathsf{V}[\mathbb{H}]}$.
- ▷ Since $\kappa = ((2^{\aleph_0})^+)^{\mathsf{V}}$ and $j(\kappa) = ((2^{\aleph_0})^+)^{\mathsf{V}[\mathbb{H}]}$, it follows that $\mathcal{H}((2^{\aleph_0})^+)^{\mathsf{V}} \prec \mathcal{H}((2^{\aleph_0})^+)^{\mathsf{V}[\mathbb{H}]}$.

Thank you for your attention!

ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다

Gracias por su atención.

Dziękuję za uwagę.

Grazie per l'attenzione.

Dank u voor uw aandacht.

Ich danke Ihnen für Ihre Aufmerksamkeit.

Proof of Lemma 7.

Lemma 7.([CC]) Suppose that $\mathcal{P}$ satisfies $(\dagger)$ and κ is $\mathcal{P}$-gen. weakly-compact. Then (1) κ is weakly Mahlo. (2) κ has the tree property.

Proof. We prove (1): Suppose that $C \subseteq \kappa$ is a club. Let $A \subseteq \kappa$ be s.t. it codes C as well as witnesses of singularity of all singular cardinals and being successor of successor cardinals $< \kappa$.

- Let M be a transitive model of ZFC^- s.t. κ, $A \in M$ and there is a ν-cc p.o. $\mathbb{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ s.t. there are j, $N \in \mathsf{V}[\mathbb{G}]$ with $j : M \overset{\prec}{\rightarrow}_{\kappa} N$.
- Note $C \in M$ by $A \in M$. We have $N \models$"$j(C)$ is a club subset of $j(\kappa)$" by elementarity. Since $j(C) \cap \kappa = C$ by $crit(j) = \kappa$, it follows that $\kappa \in j(C)$. κ is regular. Since $\mathbb{P}$ preserves cardinality and cofinality $\geq \nu$ by its ν-cc, $\mathsf{V}[\mathbb{G}] \models$"$\kappa$ is regular". It follows that $N \models$"κ is regular". Thus $N \models$"$j(C)$ contains a regular cardinal" and $M \models$"C contains a regular cardinal" by elementarity. This implies that κ is a weakly Mahlo cardinal.

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A sketch of the proof of Proposition 3.

- We imitate the ultraproduct construction:
- Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. We work in $\mathsf{V}[\mathbb{G}]$. Let
  - $\mathcal{F} := \{f \in \mathsf{V} \,:\, f : \mathrm{dom}(f) \to \mathsf{V}, \mathrm{dom}(f) \in \mathcal{H}(\theta)^{\mathsf{V}}\}$, and
  - $\Pi := \{\langle f, a\rangle \,:\, f \in \mathcal{F},\ a \in j_0(\mathrm{dom}(f))\}$.

  For $\langle f, a\rangle$, $\langle g, b\rangle \in \Pi$, let
  - $\langle f, a\rangle \sim \langle g, b\rangle \;:\Leftrightarrow\; \langle a, b\rangle \in j_0(S_{f(x)=g(y)})$, where $S_{f(x)=g(y)} := \{\langle u, v\rangle \,:\, u \in \mathrm{dom}(f),\ v \in \mathrm{dom}(g),\ f(u) = g(v)\}$;

  and
  - $\langle f, a\rangle\ E\ \langle g, b\rangle \;:\Leftrightarrow\; \langle a, b\rangle \in j_0(S_{f(x)\varepsilon g(y)})$, where $S_{f(x)\varepsilon g(y)} := \{\langle u, v\rangle \,:\, u \in \mathrm{dom}(f),\ v \in \mathrm{dom}(g),\ f(u) \in g(v)\}$.

- ① $\sim$ is a congruent relation to E; Let $\check{f}_u : \{\emptyset\} \to \{u\}$ for $u \in \mathsf{V}$, then ② $i : \mathsf{V} \to \Pi/{\sim}$; $u \mapsto \langle \check{f}_u, \emptyset\rangle/{\sim}$ is an elementary embedding; ③ $(\Pi/{\sim}, E/{\sim})$ is well-founded and set-like; and ④ The Mostowski collapse M of $(\Pi/{\sim}, E/{\sim})$ together with the canonical embedding j of V into M is as desired. ⧉

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Proof of Theorem 2.

Theorem 2. ([II]) Suppose that each element of $\mathcal{P}$ is μ-cc for some $\mu < \kappa$. If κ is $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$.

Proof. Let $\mathbb{P}$ and κ be as above.

- $2^{\aleph_0} \geq \kappa$ follows from the following Lemma:

Lemma A1. (Lemma 5.5 in [II]) Suppose that $\mathcal{P}$ is a class of p.o.s containing a p.o. $\mathbb{P}$ which adds a real. If κ is a $\mathcal{P}$-Laver-gen. supercompact, then $2^{\aleph_0} \geq \kappa$.

Proof of Lemma A1.: Let $\mathbb{P} \in \mathcal{P}$ be s.t. any generic filter over P codes a new real.

- Suppose that $\mu < \kappa$ We have to show hat $2^{\aleph_0} > \mu$.
- ▷ Let $\vec{a} = \langle a_\xi \ : \ \xi < \mu \rangle$ be a sequence of subsets of ω.
- It is enough to show that $\vec{a}$ does not enumerate $\mathcal{P}(\omega)$.
- ▷ By $\mathcal{P}$-Laver-gen. supercompactness of κ, there are $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ and $j, M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$ with $\mathbb{P}, \mathbb{H} \in M$.

Proof of Theorem 2. (2/3)

- Since $\mu < \kappa$, we have $j(\vec{a}) = \vec{a}$. Since $\mathbb{H} \cap \mathbb{P} \in M$ codes a real not in V, we have

▷ $M \models$ "$j(\vec{a})$ does not enumerate $\mathcal{P}(\omega)$".

By elementarity, it follows that

▷ V $\models$ "$\vec{a}$ does not enumerate $\mathcal{P}(\omega)$". (Lemma A1)

Proof of Theorem 2. (3/3)

Theorem 2. ([II]) Suppose that each element of $\mathcal{P}$ is μ-cc for some $\mu < \kappa$. If κ is $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$.

Continuation of the Proof of Theorem 2.: We prove $2^{\aleph_0} \leq \kappa$.

- Let $\lambda \geq \kappa$, $2^{\aleph_0}$ be sufficiently large and let $\mathbb{P} \in \mathcal{P}$, $\underset{\sim}{\mathbb{Q}}$ a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and $\mathbb{H}$ a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic set with $j, M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\preceq}{\to}_{\kappa} M$, (*) $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa) > \lambda$, $\mathbb{H}$, $\mathbb{P} \in M$, and (**) $j''j(\kappa) \in M$.
- Since κ is regular (this follows already from $\mathcal{P}$-gen. largeness of κ), $M \models$"$j(\kappa)$ is regular" by elementarity. By (**) it follows that $j(\kappa)$ is regular in $\mathsf{V}[\mathbb{H}]$. Hence it is also regular in V.
- By assumption $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ has μ-cc for some $\mu < \kappa$.
- Since the chain condition of $\mathcal{P}$ and $\mathcal{P}$-gen. supercompactness of κ implies SCH above $2^{<\kappa}$ ([II]), we have $\mathsf{V} \models$"$(j(\kappa))^{<\mu} = j(\kappa)$". By (*) it follows that $\mathsf{V}[\mathbb{G} \models$"$2^{\aleph_0} \leq j(\kappa)$". By (**) we have $(j(\kappa)^+)^M = (j(\kappa)^+)^{\mathsf{V}[\mathbb{H}]}$. Thus $M \models 2^{\aleph_0} \leq j(\kappa)$. By elementarity, it follows that $V \models 2^{\aleph_0} \leq \kappa$. □ (Theorem 2.)

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