Strong Löwenheim-Skolem Theorem of stationary logics, game reflection principles and generically supercompact cardinals

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The solution of the Continuum Problem

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  ▷ Provided that a sufficiently strong and reasonable reflection principle should hold.

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  ▷ Provided that a Laver-generically supercompact cardinal should exist. Under a Laver-generically supercompact cardinal, in each of the three scenarios, the respective reflection principle in the sense of above also holds.

The results in the following slides ...

are going to appear in joint papers with André Ottenbereit Maschio Rodriques and Hiroshi Sakai:

[1] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, submitted.
http://fuchino.ddo.jp/papers/SDLS-x.pdf

[2] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, pre-preprint. http://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[3] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, III — more on Laver-generically large cardinals, in preparation.

The size of the continuum

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The size of the continuum (1/2)

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Theorem 1. $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* CH. Proof

Actually $\mathrm{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ is equivalent with Sean Cox's Diagonal Reflection Principle for internal clubness + CH.

Theorem 2. **(a)** $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0} = \aleph_2$. Proof
(b) $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to Diagonal Reflection Principle for internal clubness* **(c)** $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *is equivalent to* $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2) + \neg\mathrm{CH}$. Proof

Theorem 3. $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0}$ *is very large (e.g. weakly Mahlo, weakly hyper Mahlo, etc.)* Proof

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Theorem 1. *If there exists a Laver-generically supercompact cardinal κ for σ-closed p.o.s, then $\kappa = \aleph_2$ and* CH *holds. Moreover* $MA^{+\aleph_1}(\sigma\text{-closed})$ *holds. Thus* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2)$ *also holds.*

Theorem 2. *If there exists a Laver-generically supercompact cardinal κ for proper p.o.s, then $\kappa = \aleph_2 = 2^{\aleph_0}$. Moreover* $\mathsf{PFA}^{+\aleph_1}$ *holds. Thus* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,2^{\aleph_0})$ *also holds.*

Theorem 3. *If there exists a Laver generically supercompact cardinal κ for c.c.c. p.o.s, then $\kappa \leq 2^{\aleph_0}$ and κ is very large (for all regular $\lambda \geq \kappa$, there is a σ-saturated normal ideal over $\mathcal{P}_\kappa(\lambda)$). Moreover* $\mathsf{MA}^{+\mu}(ccc, {<}\,\kappa)$ *for all $\mu < \kappa$ and* $\mathsf{SDLS}^{int}_+(\mathcal{L}^{PKL}_{stat}, {<}\,\kappa)$ *hold.*

Consistency of Laver-generically supercompact cardinals

Theorem 1. (1) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for* σ*-closed p.o.s" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for proper p.o.s" is consistent as well.*

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a strongly Laver-generically supercompact cardinal for c.c.c. p.o.s" is consistent as well.*

Proof. Starting from a model of ZFC with a supercompact cardinal κ (a superhuge cardinal in case of (2)), we can obtain models of respective assertions by iterating (in countable support in case of (1), (2) and in finite support in case of (3)) with respective p.o.s κ times along a Laver function (for (1) and (2) Laver function for supercompactness; for (2), Laver function for super-almost-hugeness). □

Some more background and open problems

- By a slight modification of a proof by B. König, the implication of $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ from the existence of Laver-generically supercompact cardinal for σ-closed p.o.s can be interpolated by a Game Reflection Principle which by itself characterizes the usual version of generic supercompactness of $\aleph_2$ by σ-closed p.o.s.

Problem 1. Does there exist some sort of Game Reflection Principle which plays similar role in the other two scenarios in the trichotomy?

Problem 2. Does (some variation of) Laver-generic supercompactness of κ for c.c.c. p.o.s imply $\kappa = 2^{\aleph_0}$?

Problem 3. Is there any characterization of $\mathsf{MA}^{++}(...)$ which would fit our context?

Problem 4. What is about Laver-generic supercompactness for Cohen reals? What is about Laver-generic supercompactness for stationary preserving p.o.s?

A partial solution of Problem 2

Lemma 1. Suppose that $\mathcal{P}$ is a class of p.o.s containing a p.o. $\mathbb{P}$ which adds a new real. If κ is a Laver-generically supercompact for $\mathcal{P}$, then $\kappa \leq 2^{\aleph_0}$.

Proof. Let $\mathbb{P} \in \mathcal{P}$ be s.t. any generic filter over $\mathbb{P}$ codes a new real. Suppose that $\mu < \kappa$. We show that $2^{\aleph_0} > \mu$. Let $\vec{a} = \langle a_\xi : \xi < \mu \rangle$ be a sequence of subsets of ω. It is enough to show that $\vec{a}$ does not enumerate $\mathcal{P}(\omega)$.

- By Laver-generic supercompactness of κ for $\mathcal{P}$, there are $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, transitive $M \subseteq \mathsf{V}[\mathbb{H}]$ and $j \subseteq M[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\rightarrow} M$, $crit(j) = \kappa$ and $\mathbb{P}, \mathbb{H} \in M$. Since $\mu < \kappa$, we have $j(\vec{a}) = \vec{a}$.
- Since $\mathbb{H} \in M$ where $\mathbb{G} = \mathbb{H} \cap \mathbb{P}$ and $\mathbb{G}$ codes a new real not in V, we have

  $$M \models \text{“} j(\vec{a}) \text{ does not enumerate } 2^{\aleph_0}\text{”}.$$
- By elementarity, it follows that

  $$\mathsf{V} \models \text{“} \vec{a} \text{ does not enumerate } 2^{\aleph_0}\text{”}.$$

□

A partial solution of Problem 2 (2/2)

Theorem 2. If κ is tightly Laver-generically superhuge for ccc p.o.s, then $\kappa = 2^{\aleph_0}$.

Proof. Suppose that κ is tightly Laver-generically superhuge for ccc p.o.s. By Lemma 1 on the previous slide, we have $2^{\aleph_0} \geq \kappa$.

To prove $2^{\aleph_0} \leq \kappa$, let $\lambda \geq \kappa$, $2^{\aleph_0}$ be large enough and let $\mathbb{Q}$ be a ccc p.o. s.t. there are $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and $j : \mathsf{V} \overset{\prec}{\to} M \subseteq \mathsf{V}[\mathbb{H}]$ with $crit(j) = \kappa$, $|\mathbb{Q}| \leq j(\kappa) > \lambda$, $\mathbb{H} \in M$ and $j''j(\kappa) \in M$.

- Since $M \models$ "$j(\kappa)$ is regular" by elementarity, $j(\kappa)$ is also regular in V by the closedness of M. Thus, we have $\mathsf{V} \models$ "$j(\kappa)^{\aleph_0} = j(\kappa)$" by SCH above $\max\{\kappa, 2^{\aleph_0}\}$ (available under the assumption on κ).
- Since $\mathbb{Q}$ has the ccc and $|\mathbb{Q}| \leq j(\kappa)$, it follows that $\mathsf{V}[\mathbb{H}] \models$ "$2^{\aleph_0} \leq j(\kappa)$". Now we have $(j(\kappa)^+)^M = (j(\kappa)^+)^{\mathsf{V}[\mathbb{H}]}$ by $j''j(\kappa) \in M$. Thus $M \models$ "$2^{\aleph_0} \leq j(\kappa)$".
- By elementarity, it follows that $\mathsf{V} \models$ "$2^{\aleph_0} \leq \kappa$". □

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- Since $M \models$"$j(\kappa)$ is regular" by elementarity, $j(\kappa)$ is also regular in V by the closedness of M. Thus, we have $\mathsf{V} \models$"$j(\kappa)^{\aleph_0} = j(\kappa)$" by SCH above $\max\{\kappa, 2^{\aleph_0}\}$ (available under the assumption on κ).
- Since $\mathbb{Q}$ has the ccc and $|\mathbb{Q}| \leq j(\kappa)$, it follows that $\mathsf{V}[\mathbb{H}] \models$"$2^{\aleph_0} \leq j(\kappa)$". Now we have $(j(\kappa)^+)^M = (j(\kappa)^+)^{\mathsf{V}[\mathbb{H}]}$ by $j''j(\kappa) \in M$. Thus $M \models$"$2^{\aleph_0} \leq j(\kappa)$".
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- Since $M \models$ "$j(\kappa)$ is regular" by elementarity, $j(\kappa)$ is also regular in V by the closedness of M. Thus, we have $\mathsf{V} \models$ "$j(\kappa)^{\aleph_0} = j(\kappa)$" by SCH above $\max\{\kappa, 2^{\aleph_0}\}$ (available under the assumption on κ).
- Since $\mathbb{Q}$ has the ccc and $|\,\mathbb{Q}\,| \leq j(\kappa)$, it follows that $\mathsf{V}[\mathbb{H}] \models$ "$2^{\aleph_0} \leq j(\kappa)$". Now we have $(j(\kappa)^+)^M = (j(\kappa)^+)^{\mathsf{V}[\mathbb{H}]}$ by $j''j(\kappa) \in M$. Thus $M \models$ "$2^{\aleph_0} \leq j(\kappa)$".
- By elementarity, it follows that $\mathsf{V} \models$ "$2^{\aleph_0} \leq \kappa$". □

Thank you for your attention.

A Proof of: $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$ implies $2^{\aleph_0}$ is very large.

- For a regular cardinal κ and a cardinal $\lambda \geq \kappa$, $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$ is said to be 2-*stationary* if, for any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there is an $a \in \mathcal{S}$ s.t. $|\kappa \cap a|$ is a regular uncountable cardinal and $\mathcal{T} \cap \mathcal{P}_{\kappa\cap a}(a)$ is stationary in $\mathcal{P}_{\kappa\cap a}(a)$. A regular cardinal κ has the 2-*stationarity property* if $\mathcal{P}_\kappa(\lambda)$ is 2-stationary (as a subset of itself) for all $\lambda \geq \kappa$.

Lemma 1. *For a regular uncountable κ, $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ implies that κ is 2-stationary.*

Lemma 2. *Suppose that κ is a regular uncountable cardinal.*

(1) *If κ is 2-stationary then κ is a limit cardinal.*

(2) *For any $\lambda \geq \kappa$, 2-stationary $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$, and any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there are stationarily many $r \in \mathcal{S}$ s.t. $\mathcal{T} \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.*

(3) *If κ is 2-stationary then κ is a weakly Mahlo cardinal.*

$$\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0}) \;\Leftrightarrow\; \mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2) + \neg\mathrm{CH}.$$

- If $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ holds then $2^{\aleph_0} = \aleph_2$ by (a). Thus, it follows that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2) + \neg\mathrm{CH}$ holds.
- Suppose $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ holds. Then we have $2^{\aleph_0} \leq \aleph_2$ by a theorem of Todorčević already mentioned. Thus, if $2^{\aleph_0} > \aleph_1$ in addition, we have $2^{\aleph_0} = \aleph_2$. Thus $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ follows.

□

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Baumgartner's Theorem

- $\kappa > |M| \geq |\lambda \cap M| \geq \aleph_2$
- there is a club $C \subseteq [M]^{\aleph_0}$ with $C \subseteq M$

Theorem 1 (J.E. Baumgartner). *Let $\omega < \kappa < \lambda$ and κ be regular. Then any club subset of $[\lambda]^{<\kappa}$ has cardinality $\geq \lambda^{\aleph_0}$.*

- $\kappa > |M| \geq |C| \geq 2^{\aleph_0}$.

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$\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ for $\kappa > \aleph_2$ implies $\kappa > 2^{\aleph_0}$.

- $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies $2^{\aleph_0} \leq \aleph_2$: it is easy to see that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the reflection principle $\mathrm{RP}(\omega_2)$ in Jech's [millennium-book]. $\mathrm{RP}(\omega_2)$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević).
  - It follows that $\kappa > \aleph_2 \geq 2^{\aleph_0}$.
- Thus, we may assume that $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ does not hold. Hence there is a structure $\mathfrak{A}$ s.t., for any $\mathfrak{B} \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}$, we have $\|\mathfrak{B}\| \geq \aleph_2$. Let $\lambda = \|\mathfrak{A}\|$. W.l.o.g., we may assume that the underlying set of $\mathfrak{A}$ is λ. Let $\mathfrak{A}^* = \langle \mathcal{H}(\lambda^+), \underbrace{\lambda, ...,}_{=\mathfrak{A}} \in \rangle$.
- By $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$, there is $M \in [\mathcal{H}(\lambda^+)]^{<\kappa}$ s.t. $\mathfrak{A}^* \restriction M \prec^-_{callL^{\aleph_0}_{stat}} \mathfrak{A}^*$. It follows that $\mathfrak{A} \restriction (\lambda \cap M) \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}$. By the choice of $\mathfrak{A}$, we have $|M| \geq |\lambda \cap M| \geq \aleph_2$.
- By elementarity, there is $C \subseteq [M]^{\aleph_0} \cap M$ which is a club in $[M]^{\aleph_0}$. By a theorem of Baumgartner, it follows that $\kappa > |M| \geq |C| \geq 2^{\aleph_0}$. □

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ implies $2^{\aleph_0} = \aleph_2$.

Proposition 1. $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ *for* $\kappa > \aleph_2$ *implies* $\kappa > 2^{\aleph_0}$.

Proof

- Suppose that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds. Then $2^{\aleph_0} \leq \aleph_2$ by the Proposition 1.
- $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_1)$ does not hold since

  "there are uncountably many x s.t. ..."

  is expressible in $\mathcal{L}^{\aleph_0}_{stat}$. $\Big[$ e.g. by $stat\, X\, (\exists x\, (\cdots \ \wedge \ x \not\in X))$ $\Big]$

  Thus, $2^{\aleph_0} > \aleph_1$. □

Corollary 2. $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ *is inconsistent.*

Proof. Assume $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$. Then $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ holds. Thus $2^{\aleph_0} = \aleph_2$ by the proof above. But then $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ holds. By Proposition 1. This implies $2^{\aleph_0} = \aleph_0$. This is a contradiction. □

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SDLS($\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2$) implies CH.

- Suppose that $\mathfrak{A} = \langle \mathcal{H}(\omega_1), \in \rangle$ and Let $B \in [\mathcal{H}(\omega_1)]^{<\aleph_2}$ be s.t. $\mathfrak{A} \restriction B \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. Then for any $U \in [B]^{\aleph_0}$ we have $\mathfrak{A} \models$ "$\exists x \forall y \, (y \in x \leftrightarrow y \, \varepsilon \, U)$".
- By elementarity we also have $\mathfrak{B} \models$ "$\exists x \forall y \, (y \in x \leftrightarrow y \, \varepsilon \, U)$".
- ▷ It follows that $U \in B$. Thus $[B]^{\aleph_0} \subseteq B$ and $2^{\aleph_0} \leq |\, B \,| \leq \aleph_1$. □

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Strong Downward Löwneheim-Skolem Theorem for stationary logic

- $\mathcal{L}^{\aleph_0}_{stat}$ is a weak second order logic with monadic second-order variables X, Y etc. which run over the countable subsets of the underlying set of a structure. The logic has only the weak second order quantifier "*stat*" and its dual "*aa*" (but not the second-order existential (or universal) quantifiers) with the interpretation:

  $\mathfrak{A} \models stat\, X\, \varphi(\ldots, X) \quad :\Leftrightarrow$
  $\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(\ldots, U)\}$ is a stationary subset of $[A]^{\aleph_0}$.

- For $\mathfrak{B} = \langle B, \ldots \rangle \subseteq \mathfrak{A}$, $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models \varphi(a_0, \ldots, U_0, \ldots) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, \ldots, U_0, \ldots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \ldots, X_0, \ldots)$ and for all $a_0, \ldots \in B$ and for all $U_0, \ldots \in [B]^{\aleph_0}$.

- SDLS$(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \quad :\Leftrightarrow$
  For any structure $\mathfrak{A} = \langle A, \ldots \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

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A weakening of the Strong Downward Löwneheim-Skolem Theorem

- For $\mathfrak{B} = \langle B, ... \rangle \subseteq A$, $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ $\;:\Leftrightarrow$
  $\mathfrak{B} \models \varphi(a_0, ...) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, ...)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ...)$ without free seond-order variables and for all $a_0, ... \in B$.

- $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ $\;:\Leftrightarrow$
  For any structure $\mathfrak{A} = \langle A, ... \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

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Strong Downward Löwneheim-Skolem Theorem for PKL logic

- $\mathcal{L}^{PKL}_{stat}$ is the weak second-order logic with monadic second order variables X, Y, etc. with built-in unary predicate symbol $\underbrace{K}$. The monadic seond order variables run over elements of $\mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A)$ for a structure $\mathfrak{A} = \langle A, \underbrace{K}^{\mathfrak{A}}, \ldots\rangle$ where we denote $\mathcal{P}_S(T) = \mathcal{P}_{|\,S\,|}(T) = \{u \subseteq T \ : \ |\,u\,| < |\,S\,|\}$. The logic has the unique second order quantifier "*stat*" (and its dual).

- The internal interpretation of the quantifier is defined by:

  $\mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, \ldots, U_0, \ldots, X) \quad :\Leftrightarrow$
  $\{U \in \mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A) \cap A \ : \ \mathfrak{A} \models^{int} \varphi(a_0, \ldots, U_0, \ldots, U)\}$ is a stationary subset of $\mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A)$ for $a_0, \ldots A$ and $U_0, \ldots \in \mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A) \cap A$.

- For $\mathfrak{B} = \langle B, K \cap B, \ldots\rangle \subseteq \mathfrak{A} = \langle A, K, \ldots\rangle$, $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models^{int} \varphi(a_0, \ldots, U_0, \ldots) \Leftrightarrow \mathfrak{A} \models^{int} \varphi(a_0, \ldots, U_0, \ldots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \ldots)$ $a_0, \ldots \in B$ and $U_0, \ldots \in \mathcal{P}_{K\cap B}(B) \cap B$.

Strong Downward Löwneheim-Skolem Theorem for PKL logic (2/2)

- $\mathrm{SDLS}^{int}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ $\quad :\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, \dots \rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$. $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

- $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ $\quad :\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, \dots \rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$. $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$, there are stationarily many structures $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

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Laver generically supercompact cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a Laver-generically supercomact for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

  (1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

  (2) $\mathbb{P}, \mathbb{H} \in M$,

  (3) $j''\lambda \in M$.

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tightly Laver generically superhuge cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a tightly Laver-generically superhuge for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

  (1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

  (2) $\mathbb{P}, \mathbb{H} \in M$,

  (3) $j''j(\kappa) \in M$, and

  (4) $|\mathbb{Q}| \leq j(\kappa)$.

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Diagonal Reflection Principle

- (S. Cox) For a regular cardinal $\theta > \aleph_1$:

  DRP(θ, IC): There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ s.t.

  (1) $M \cap \mathcal{H}(\theta)$ is internally club;

  (2) for all $R \in M$ s.t. R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

- For a regular cardinal $\lambda > \aleph_1$

  $(*)_\lambda$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$, if $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$, is a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$, then there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, for all $a \in M$.

Proposition 1. *TFAE:* (a) *The global version of Diagonal Reflection Principle of S.Cox for internal clubness (i.e. DRP(θ, IC) for all regular $\theta > \aleph_1$) holds.*

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(c) $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *holds.*

Reflection Principles $\mathsf{RP}_{??}$

- The following are variations of the "Reflection Principle" in [Jech, Millennium Book].

$\mathsf{RP}_{\mathsf{IC}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

$\mathsf{RP}_{\mathsf{IU}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally unbounded $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

Since every internally club M is internally unbounded, we have:

Lemma 1. $\mathsf{RP}_{\mathsf{IC}}$ *implies* $\mathsf{RP}_{\mathsf{IU}}$*.*

$\mathsf{RP}_{\mathsf{IU}}$ is also called Axiom R in Set-Theoretic Topology.

Theorem 2. ([Fuchino, Juhasz etal. 2010]) $\mathsf{RP}_{\mathsf{IU}}$ *implies* FRP*.*

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Stationary subsets of $[X]^{\aleph_0}$

- $C \subseteq [X]^{\aleph_0}$ is club in $[X]^{\aleph_0}$ if (1) for every $u \in [X]^{\aleph_0}$, there is $v \in C$ with $u \subseteq v$; and (2) for any countable increasing chain $\mathcal{F}$ in C we have $\bigcup \mathcal{F} \in C$.

▷ $S \subseteq [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$ if $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^{\aleph_0}$.

- A set M is internally unbounded if $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$ (w.r.t. $\subseteq$)

▷ A set M is internally stationary if $M \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$

▷ A set M is internally club if $M \cap [M]^{\aleph_0}$ contains a club in $[M]^{\aleph_0}$.

"Diagonal Reflection Principle" にもどる

"$\mathrm{RP}_{??}$" にもどる

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\kappa_\omega$ and any mapping $g : E \to [\kappa]^{\aleph_0}$ with $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$, there is $\gamma \in E^\kappa_{\omega_1}$ s.t.

(*) for any $I \in [\gamma]^{\aleph_1}$ closed w.r.t. g and club in γ, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$.

- ▷ $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |\, I \,|$ s.t. $I = \bigcup_{\alpha<\lambda} I_\alpha$.
- ▶ FRP follows from Martin's Maximum or Rado's Conjecture. MA^+(σ-closed) already implies FRP but PFA does not imply FRP since PFA does not imply stationary reflection of subsets of $E^{\omega_2}_\omega$ (Magidor, Beaudoin) which is a consequence of FRP.
- ▶ FRP is a large cardinal property: FRP implies the total failure of the square principle.
- ▷ FRP is known to be equivalent to the reflection of uncountable coloring number of graphs down to cardinality $< \aleph_2$.

Proof of Fact 1

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal κ there is a non-metrizable space X of size κ s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $\kappa' \geq \kappa$ be of cofinality $\geq \kappa, \ \omega_1$.
- The topological space $(\kappa' + 1, \mathcal{O})$ with

$$\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} \ : \ x \subseteq \kappa', \ x \text{ is bounded in } \kappa'\}$$

  is non-metrizable since the point κ' has character $= \mathrm{cf}(\kappa') > \aleph_0$.
- Any subspace of $\kappa' + 1$ of size $< \kappa$ is discrete and hence metrizable.

□

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Proof of Fact 3

- It is enough to prove the following:

Lemma 1. (Folklore ?, see [Fuchino, Juhasz etal. 2010]) *For a regular cardinal $\kappa \geq \aleph_2$ if, there is a non-reflectingly stationary $S \subseteq E^{\kappa}_{\omega}$, then there is a non meta-lindelöf (and hence non metrizable) locally compact and locally countable topological space X of cardinality κ s.t. all subspace Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $I = \{\alpha + 1 : \alpha < \kappa\}$ and $X = S \cup I$.
- Let $\langle a_\alpha : \alpha \in S\rangle$ be s.t. $a_\alpha \in [I \cap \alpha]^{\aleph_0}$, a_α is of order-type ω and cofinal in α. Let $\mathcal{O}$ be the topology on X introduced by letting

  (1) elements of I are isolated; and

  (2) $\{a_\alpha \cup \{\alpha\} \setminus \beta : \beta < \alpha\}$ a neighborhood base of each $\alpha \in S$.

- Then $(X, \mathcal{O})$ is not meta-lindelöf (by Fodor's Lemma) but each $\alpha < \kappa$ as subspace of X is metrizable (by induction on α).□ もどる

Coloring number and chromatic number of a graph

- For a cardinal $\kappa \in \mathrm{Card}$, a graph $G = \langle G, K \rangle$ has coloring number $\leq \kappa$ if there is a well-ordering $\sqsubseteq$ on G s.t. for all $p \in G$ the set

$$\{q \in G \ : \ q \sqsubseteq p \text{ and } q \, K \, p\}$$

has cardinality $< \kappa$.

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- The coloring number $col(G)$ of a graph G is the minimal cardinal among such κ as above.
- The chromatic number $chr(G)$ of a graph $G = \langle G, K \rangle$ is the minimal cardinal κ s.t. G can be partitioned into κ pieces $G = \bigcup_{\alpha<\kappa} G_\alpha$ s.t. each G_α is pairwise non adjacent (independent).
- For all graph G we have $chr(G) \leq col(G)$.

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κ-special trees

- For a cardinal κ, a tree T is said to be κ-special if T can be represented as a union of κ subsets T_α, $\alpha < \kappa$ s.t. each T_α is an antichain (i.e. pairwise incomparable set).

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Stationary subset of E^κ_ω

- For a cardinal κ,

$$E^\kappa_\omega = \{\gamma < \kappa : \mathrm{cf}(\gamma) = \omega\}.$$

- A subset $C \subseteq \xi$ of an ordinal ξ of uncountable cofinality, C is closed unbounded (club) in ξ if (1): C is cofinal in ξ (w.r.t. the canonical ordering of ordinals) and (2): for all $\eta < \xi$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.

- $S \subseteq \xi$ is stationary if $S \cap C \neq \emptyset$ for all club $C \subseteq \xi$.

- A stationary $S \subseteq \xi$ if reflectingly stationary if there is some $\eta < \xi$ of uncountable cofinality s.t.$S \cap \eta$ is stationary in η. Thus:

- A stationary $S \subseteq \xi$ if non reflectingly stationary if $S \cap \eta$ is non stationary for all $\eta < \xi$ of uncountable cofinality.

Proof of Theorem 1.

CH $\Rightarrow$ SDLS($\mathcal{L}^{\aleph_0,II}, <\aleph_2$): For a structure $\mathfrak{A}$ with a countable signature L and underlying set A, let θ be large enough and $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\theta), A, \mathfrak{A}, \in \rangle$. where $A = \underset{\sim}{A}^{\tilde{\mathfrak{A}}}$ for a unary predicate symbol $\underset{\sim}{A}$ and $\mathfrak{A} = \underset{\sim}{\mathfrak{A}}^{\tilde{\mathfrak{A}}}$ for a constant symbol $\underset{\sim}{\mathfrak{A}}$. Let $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ be s.t. $|B| = \aleph_1$ for the underlying set B of $\mathfrak{B}$ and $[B]^{\aleph_0} \subseteq B$.
$\mathfrak{B} = \mathfrak{A} \restriction \underset{\sim}{A}^{\tilde{\mathfrak{B}}}$ is then as desired.

SDLS($\mathcal{L}^{\aleph_0}, <\aleph_2$) $\Rightarrow$ CH: Suppose $\mathfrak{A} = \{\omega_2 \cup [\omega_2]^{\aleph_0}, \in\}$. Consider the $\mathcal{L}^{\aleph_0}$-formula $\varphi(X) = \exists x \forall y\, (y \in x \leftrightarrow y \,\varepsilon\, X)$.
If $\mathfrak{B} = \langle B, ... \rangle$ is s.t. $|B| \leq \aleph_1$ and $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}}$, then for $C \in [B]^{\aleph_0}$, since $\mathfrak{A} \models \varphi(C)$, we have $\mathfrak{B} \models \varphi(C)$. It dollows that $[B]^{\aleph_0} \subseteq B$ and $2^{\aleph_0} \leq (|B|)^{\aleph_0} \leq |B| = \aleph_1$.