Set-theoretic results in mathematics

Sakaé Fuchino (**渕野 昌**)

Graduate School of System Informatics
Kobe University
(神戸大学大学院 システム情報学研究科)

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

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Incompleteness of mathematics

The 1st Incompleteness Theorem (K. Gödel, 1931)
For any (concretely given) consistent axiom system, in which the elementary arithmetic can be developed, there is an assertion in elementary arithmetic which can neither proved nor disproved from the axiom system.

The 2nd Incompleteness Theorem (K. Gödel, 1931)
In any (concretely given) consistent axiom system T, in which the elementary arithmetic can be developed, the assertion "T is consistent" (formulated in terms of "coding" of the logical formulas and proofs in natural numbers) cannot be proved.

Incompleteness of mathematics (2/2)

- All (conventional) mathematical theories and their proofs can be (re)formulated in the (axiomatic) set theory. — At least we do not know any counter example to this claim.

▷ Incompleteness Theorems also apply to the axiom system of set theory (ZFC). Thus:

- Mathematics is not complete. I.e., there are assertions (in the language of mathematics) which can be neither proved nor disproved by any conventional mathematical argument.
- There is no ultimate guarantee that mathematics is consistent. — There are many "partial" guarantees of the consistency of mathematics!

▷ For an axiom system T, an assertion φ in the language of T is **consistent with** T if $\neg\varphi$ (the negation of φ) is not provable from T. ▷ φ is said to be **independent from** T if both φ and $\neg\varphi$ are not provable from T. ▷ Note that we can sometimes **prove** the consistency or even independence of some φ over T like in the proof of the Incompleteness Theorems!

Continuum Problem

- The arithmetical assertions constructed in the proof of the Incompleteness Theorems and proved there to be independent form a given theory were quite "artificial" ones.
- Historically, **Continuum Hypothesis** is one of the first examples of "mathematical" assertions proved to be independent from the usual axiom system of set theory (ZFC).
- A linear ordering $\sqsubseteq$ on a class X is said to be **well-ordering** if every initial segment of X w.r.t. $\sqsubseteq$ is a set and, for any set $s \subseteq X$, s contains the minimal element w.r.t. $\sqsubseteq$.
- Cardinals (sets representing each possible size of infinite sets) are well-ordered (according to their size) and they are called $\aleph_0 < \aleph_1 < \aleph_2 < \cdots \aleph_\omega \le \aleph_{\omega+1} < \cdots$.

  $\aleph_0$: the countable cardinal

  $\aleph_1$: the first uncountable cardinal $\cdots$
- The cardinality (the cardinal representing the infinite size) of the continuum (the set of all real numbers $\mathbb{R}$) is denoted by $2^{\aleph_0}$.

Continuum Problem (2/3)

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▷ The cardinality (the cardinal representing the infinite size) of the continuum (the set of all real numbers $\mathbb{R}$) is denoted by $2^{\aleph_0}$.

▶ G. Cantor proved $2^{\aleph_0} \geq \aleph_1$ $(> \aleph_0)$ on 7. November 1873.

Continuum Hypothesis. $2^{\aleph_0} = \aleph_1$.

Theorem. (P. Cohen, 1963) Continuum Hypothesis is independent from the standard axiom system of set theory (ZFC).

Continuum Problem (3/3)

Continuum Hypothesis. $2^{\aleph_0} = \aleph_1$.

Theorem. (P. Cohen, 1963) The Continuum Hypothesis (CH) is independent from the standard axiom system of set theory (ZFC).

- This result should not be considered as the final solution to the continuum problem. ▷ It was rather a beginning of a new branch of mathematics!
- ▷ The independency of the Continuum Hypothesis can be interpreted as it suggests that some axioms of set theory are still missing which would decide among other things the size of the continuum.
- ▷ Today we have several candidates of additional axioms — some of them imply the Continuum Hypothesis while some other $2^{\aleph_0} = \aleph_2$.

Set-theoretic multiverse

- The independence proof of Cohen introduced the method of **forcing** with which we can construct extended models (generic extensions) of a given model of set theory — Cohen's independence proof of CH corresponds to the construction of generic extensions in which $2^{\aleph_0} = \aleph_1$ (or $2^{\aleph_0} \neq \aleph_1$) holds.

- The method of forcing enables a study of the correlations between mathematical assertions which are proved to be independent over ZFC.

- The plenitude of diverse models obtained by the forcing construction suggests also the vantage point which is recently named "**set-theoretic multiverse**" in which we regard the entirety of the possible models of set theory — the class of set-theoretic possible worlds — as the ultimate objective of the set theory.

Some examples of independent "mathematical" assertions

- In the rest of the talk, I shall give some known results in connection with independent mathematical assertions.
- We shall discuss about:

▷ Whitehead problem (proved to be independent by S. Shelah, 1974)

▷ The simplicity of the automorphism group of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ (proved to be independent by S. Shelah, (D. v.Douwen) and S.F., 1991)

▷ A generalization of Helly's theorem for monotone functions (which is characterized by a cardinal invariant whose value is independent — i.e. can not be decided in ZFC, S.F. and Sy. Plewik, 1999)

▷ A reflection theorem of metrizability of locally compact spaces (proved to be independent by Z. Balogh in 2001, exact set-theoretic characterization given by S.F., L. Soukup, H. Sakai et al. 2010, 201?)

Whitehead Problem

- For groups A, B, a surjective homomorphism $\pi : B \to A$ **splits** if there is a homomorphism $\rho : A \to B$ s.t. $\pi \circ \rho = 1_A$.

Fact. A group A is free if and only if every surjective homomorphism from a group B to A splits.

- A group A is said to be a **Whitehead group** if for any surjective homomorphism $\pi : B \to A$ split if $Ker(\pi) \cong \mathbb{Z}$.
- By the Fact above every free group is Whitehead.

Are all Whitehead group free?

Theorem. (S.Shelah, 1974) The assertion "all Whitehead group are free" is independent from ZFC. The assertion is even independent from $\mathrm{ZFC} + \mathrm{CH}$.

- For more details see e.g.: P.C. Eklof, Whiteheads Problem is Undecidable, American Math. Monthly, Vol.83, (1979), 775–788.

Simplicity of the automorphism group of $\mathcal{P}(\mathbb{N})/fin$

- Let $\sim$ be the equivalence relation on $\mathcal{P}(\mathbb{N}) = \{X : X \subseteq \mathbb{N}\}$ defined by

  $X \sim Y \;\Leftrightarrow\; X \setminus Y$ and $Y \setminus X$ are finite.

- For each $X \in \mathcal{P}(\mathbb{N})$, let $[X]$ be its equivalence class modulo $\sim$ and

  $\mathcal{P}(\mathbb{N})/fin = \{[X] : X \in \mathcal{P}(\mathbb{N})\}$.

- For $[X], [Y] \in \mathcal{P}(\mathbb{N})/fin$, let

  $[X] \subseteq^* [Y] \;\Leftrightarrow\; X \setminus Y$ is finite.

- ▷ $\subseteq^*$ is well-defined and $(\mathcal{P}(\mathbb{N})/fin, \subseteq^*)$ is a Boolean algebra corresponding to the Stone-Čech remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of the discrete space $\mathbb{N}$.

- For many homogeneous Boolean algebras A it is known that the automorphism group $Aut(A)$ is simple (e.g. this is the case if A is σ-complete). ▷ Thus the following is a natural question:

  Is $Auto(\mathcal{P}(\mathbb{N})/fin)$ simple?

Simplicity of the automorphism group of $\mathcal{P}(\mathbb{N})/fin$ (2/3)

Is $Auto(\mathcal{P}(\mathbb{N})/fin)$ simple?

Theorem. (S. Shelah, 1982 + D. v.Douwen, 198?)

It is consistent with ZFC that $Aut(\mathcal{P}(\mathbb{N})/fin)$ is not simple.

Theorem. (S.F., 1988)

$Aut(\mathcal{P}(\mathbb{N})/fin)$ is simple under CH. Furthermore the assertion "$Aut(\mathcal{P}(\mathbb{N})/fin)$ is simple" is also consistent with $\neg\mathrm{CH}$.

- By the theorems above the simplicity of $Aut(\mathcal{P}(\mathbb{N})/fin)$ is independent from ZFC. It is even independent from $\mathrm{ZFC} + \neg\mathrm{CH}$.

Simplicity of the automorphism group of $\mathcal{P}(\mathbb{N})/fin$ (3/3)

- The following are ingredients of the proof of the theorems on the previous slide:
- An automorphism $\overline{f}$ of $\mathcal{P}(\mathbb{N})/fin$ is said to be **almost trivial** if there is a bijection f from a cofinite subset $\mathbb{N} \setminus s_0$ of $\mathbb{N}$ to another cofinite subset $\mathbb{N} \setminus s_1$ of $\mathbb{N}$ s.t. $\overline{f}([x]) = [f(x)]$ for all $x \in \mathcal{P}(\mathbb{N})$.
  - S. Shelah proved that it is consistent with ZFC that all automorphisms of $\mathcal{P}(\mathbb{N})/fin$ are almost trivial.
  - V.Douwen noticed that, if all automorphisms of $\mathcal{P}(\mathbb{N})/fin$ are almost trivial then $\{\overline{f} : f : \mathbb{N} \to \mathbb{N} \text{ is a bijection}\}$ is a normal subgroup of $Aut(\mathcal{P}(\mathbb{N})/fin)$.
- (R.D. Anderson 1968) If A is a homogeneous Boolean algebra with the following property, then $Aut(A)$ is simple: for any $a_0 \in A$ with $0 < a_0 < 1$ and $g \in Aut(A \restriction a_0)$, there are $h, \rho \in Aut(A)$ s.t.
  (a) $g \subseteq h$, (b) $\mathrm{supp}(h) \leq -\rho(a_0)$,
  (c) $\rho h(a) = h\rho(a)$ for all $a \leq -a_0$.

A generalization of Helly's theorem for monotone functions

Theorem. (E. Helly, 1921)
Any bounded sequence $\langle f_n : n \in \mathbb{N} \rangle$ of monotone real functions has a pointwise convergent subsequence $\langle f_n : n \in I \rangle$.

- Helly's Theorem has the following generalization in terms of the so-called splitting number $\mathfrak{s}$. ▷ The generalization is optimal in that it characterizes the splitting number.

Generalized Helly's Theorem. (S.F. and Sy. Plewik, 1999)
For linearly ordered sets X and Y, if Y is sequentially compact with density strictly less than $\mathfrak{s}$, any sequence of monotone functions from X to Y contains a pointwise convergent subsequence.

A generalization of Helly's theorem for monotone functions (2/3)

Generalized Helly's Theorem. (S.F. and Sy. Plewik, 1999)
For linearly ordered sets X and Y, if Y is sequentially compact with density strictly less than $\mathfrak{s}$, any sequence of monotone functions from X to Y contains a pointwise convergent subsequence.

- A linearly ordered set X is **sequentially compact** if any monotone sequence of elements of X converges in X.
- The **density** of a linearly ordered set X is the minimal size (cardinality) of a subset of X which is dense in X.
- A family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ is said to be **splitting** if, for any infinite $x \subseteq \mathbb{N}$, there is $a \in \mathcal{F}$ s.t. both $x \cap a$ and $x \setminus a$ are infinite.
- ▷ The **splitting number** $\mathfrak{s}$ is the smallest possible cardinality of a splitting family.
- It is easy to see that $\aleph_1 \leq \mathfrak{s} \leq 2^{\aleph_0}$. Thus, under CH, we have $\aleph_1 = \mathfrak{s} = 2^{\aleph_0}$. ▷ It is known that each of the equations $\aleph_1 = \mathfrak{s} < 2^{\aleph_0}$, $\aleph_1 < \mathfrak{s} = 2^{\aleph_0}$, $\aleph_1 < \mathfrak{s} < 2^{\aleph_0}$ is consistent with ZFC.

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- The generalization is optimal in that it characterizes the splitting number.
- It is easy to see that $\aleph_1 \leq \mathfrak{s} \leq 2^{\aleph_0}$. Thus, under CH, we have $\aleph_1 = \mathfrak{s} = 2^{\aleph_0}$.
  - It is known that each of the equations $\aleph_1 = \mathfrak{s} < 2^{\aleph_0}$, $\aleph_1 < \mathfrak{s} = 2^{\aleph_0}$, $\aleph_1 < \mathfrak{s} < 2^{\aleph_0}$ is consistent with ZFC.
- By the Theorem and the facts above we obtain many independent assertions like:

For linearly ordered sets X and Y, if Y is sequentially compact with density strictly less than $2^{\aleph_0}$, any sequence of monotone functions from X to Y contains a pointwise convergent subsequence.

Reflection of non-metrizability

Theorem. (Alan Dow, 1988) If an uncountable compact space X is non-metrizable then there is a non-metrizable subspace of X of cardinality $\aleph_1$.

Very rough sketch of the proof: Take sufficiently closed (more precisely: internally unbounded) elementary submodel $M \prec \mathcal{H}(\chi)$ of cardinality $\aleph_1$ with $X \in M$. Then $X \cap M$ is non metrizable. □

Theorem. (Folklore?) For any regular cardinal κ there is a topological space X which is not metrizable but all subspaces of X of cardinality $< \kappa$ are metrizable.

Proof: Let $X = \kappa + 1$ where κ is discrete and $\{\kappa + 1 \setminus \alpha : \alpha \in \kappa\}$ forms the nbhd base of κ. □

Does Dow's Theorem also hold for locally compact spaces?

Reflection of non-metrizability (2/4)

Does Dow's Theorem also hold for locally compact spaces?

Theorem. The assertion "If an uncountable locally compact space X is non-metrizable then there is a non-metrizable subspace of X of cardinality $\aleph_1$" is independent from ZFC. This assertion is independent from $\mathrm{ZFC} + \mathrm{CH}$ but also from $\mathrm{ZFC} + \neg\mathrm{CH}$. Actually this assertion is consistent with arbitrarily large continuum.

- Gödels model consisting of constructible sets provides a counter example (Folklore).
- Axiom R (a consequence of Martin's Maximum) implies the reflection of non-metrizability for locally compact spaces (Z. Balogh, 2002).
- Cohen's original construction of models for $\neg\mathrm{CH}$ preserves the assertion (S.F, L. Soukup et al. 2010).

Reflection of non-metrizability (3/4)

- The reflection (down to size $\aleph_1$) of non-metrizability for locally compact spaces can be characterized by a set-theoretic principle called FRP (Fodor-type reflection principle):

FRP: For any regular uncountable κ, for any stationary $S \subseteq \kappa$ consisting of ordinals of cofinality ω and for any $g : S \to [\kappa]^{\leq\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t.

(1) $cf(\sup I) = \omega_1$

(2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$

(3) for any regressive $f : S \cap I \to \kappa$ with $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Reflection of non-metrizability (4/4)

FRP: For any regular uncountable κ, for any stationary $S \subseteq \kappa$ consisting of ordinals of cofinality ω and for any $g : S \to [\kappa]^{\leq\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t.

(1) $cf(\sup I) = \omega_1$

(2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$

(3) for any regressive $f : S \cap I \to \kappa$ with $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Theorem. (S.F., I. Juhász, L. Soukup, Z. Szenttmiklóssy and T. Usuba, 2010)
FRP implies the reflection of non-metrizability of a locally compact space down to a subspace of cardinality $\leq \aleph_1$.

Theorem. (S.F., H. Sakai, L. Soukup and T.Usuba) The reflection in Theorem 3 implies FRP.

FRP is equivalent to many "mathematical" statements

- FRP is equivalent to the following assertions over ZFC:

  - For every uncountable locally compact space X, if X is non-metrizable then there is a non-metrizable subspace of X of carinality $\aleph_1$.
  - If an uncountable T_1-space X is not left separated then there is a subspace of X of cardinality $\aleph_1$ which is not left separated.
  - For any graph G if the coloring number of G is uncountable then there is a subgraph of G of cardinality $\aleph_1$ with uncountable coloring number.
  - If an uncountable Boolean algebra B is not openly generated then there are stationarily many subalgebras of B of cardinality $\aleph_1$ which are not openly generated (SF+A.Rinot, 2011).

Concluding Remarks

- There are “mathematical” assertions which can be proved to be independent from the set theory (and hence from the mathematics).
- If a mathematical assertion is proved to be independent form set theory, it does not necessarily mean a dead end to the theory around the assertion.
- Independence proofs can help getting deeper mathematical understanding. They can also contribute to the developments of theories which themselves do not involve independency.

Thank you for your attention.