Universality and Applications of Mathematics: a sojourn into graph theory

Crossing Bridges

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The seven bridges of Königsberg

Königsberg (present-day Kaliningrad) in 1730's

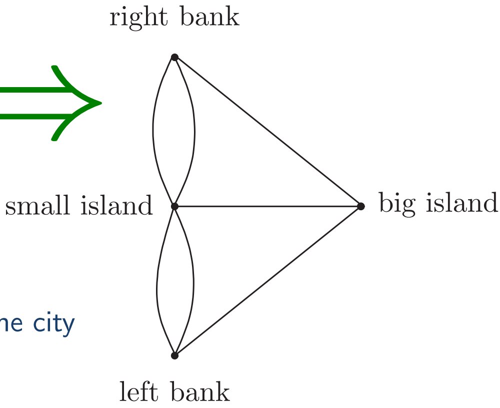
August 26, 1735 (享保(きょうほう)20 年)

- In the city of Koenigsberg (also written as "Königsberg"), there were 7 bridges connecting the banks of the river Pregel and two islands in the river, which made up the center of the city.
  Whether you can walk through the city by crossing each of the bridges exactly once was a famous riddle of the city.

- The Swiss mathematician Leonhard Euler (who was a researcher of the Academy of Science in Saint Petersburg at the time) solved this riddle negatively proving that no such stroll is possible.

Translation of the problem into a problem on a graph

$\Longleftrightarrow$



Is it possible to walk through the city of Königsberg crossing each of the 7 bridges exactly once ?

Can this diagram (graph) be drawn in a single stroke?

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A theorem of Euler

- A point x in a graph G is said to be of degree n if there are n lines connected to this point. (odd: a. 奇数の even: a. 偶数の)

Theorem 1 (Euler, 1736). Suppose that a graph G can be drawn in a single stroke. Then either (1) the degree of each point of the graph is an even number, or (2) there are exactly two points of the graph with odd degree.

Proof. By hand-waving. $\square$

The theorem above can be reformulated in contraposition as below:

Theorem 1' (Euler, 1736, reformulated). Suppose that a graph G has more than two points with odd degree, then G cannot be drawn in one stroke.

The walk is impossible!

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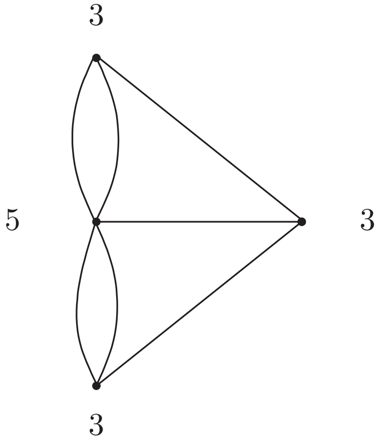

Crossing each bridge twice

- The converse of the theorem of Euler is also true;

Theorem 2 (Carl Hierholzer, 1873 (明治 6 年)). Suppose that a connected graph G is s.t. either (1) all points of G have even degree, or (2) exactly two points have odd degree. Then G can be drawn in a single stroke.

Proof. By induction on the number of lines in a graph. □

Corollary 3. In any city (whether it is Tokyo, Osaka, Kobe, New York or Paris) it is possible to walk through the city crossing each of the bridges of the city exactly twice.

Crossing each bridge twice

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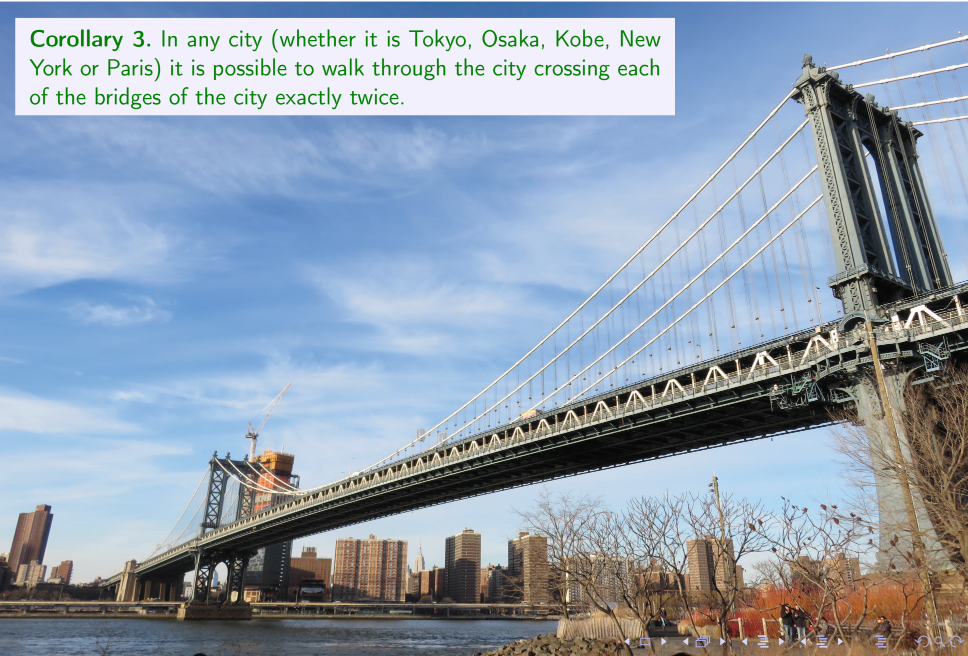


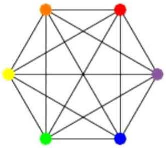
Crossing each bridge twice

Corollary 3. In any city (whether it is Tokyo, Osaka, Kobe, New York or Paris) it is possible to walk through the city crossing each of the bridges of the city exactly twice.

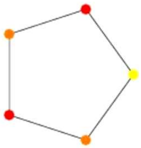
Good colorings of graphs

Good colorings of graphs

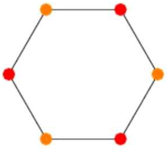
- Let us call a coloring of points of a graph G "good" if each adjacent points in the graph obtain different colors. Minimal number of colors needed for a good coloring of a graph G is called chromatic number of G and denoted by $\chi(G)$.



$\chi(G) = 6$



$\chi(G) = 3$



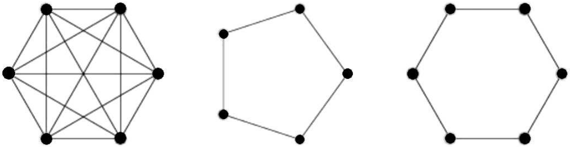
$\chi(G) = 2$

- Some infinite graphs (with infinite number of points and lines) can have finite chromatic number. This is also the case for the so called unit distance graph of the plane.

Chromatic number of the unit distance graph of the plane

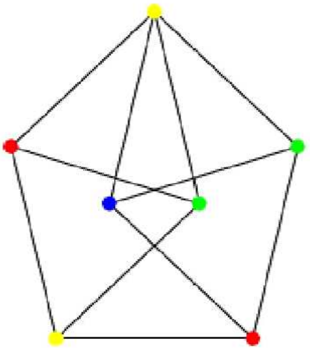
- The points of the unit distance graph of the plane $\mathcal{U}$ are all points on the two dimensional space $\mathbb{R}^2$. Two points of $\mathcal{U}$ are thought to be adjacent if and only if their distance is exactly 1.

- Of the three graphs on the last slide, the left one is not embeddable into $\mathcal{U}$ as a subgraph (since not all of the lines can be set to be of the same length) but the other two can be embedded into $\mathcal{U}$ as subgraphs.



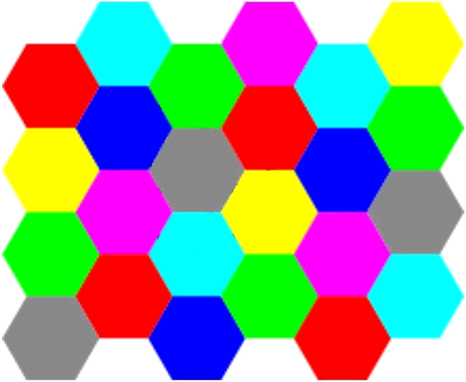
- Since the middle graph has the chromatic number 3 it follows that $\chi(\mathcal{U}) \geq 3$.

- The following graph (known as "Moser spindle") is also embeddable into the unit distance graph of the plane $\mathcal{U}$:



  - Since the Moser spindle has chromatic number 4, it follows that $\chi(\mathcal{U}) \geq 4$.

- We have also the following upper bound for $\chi(\mathcal{U})$. Consider the tiling of the plane by hexagons of diameter slightly less than 1. The hexagons can be colored in 7 colors in such a way that each neighboring hexagons obtain different colors and also two hexagons having a common neighboring hexagons are colored in different colors:



- Clearly this coloring is nice for $\mathcal{U}$.

  This shows that $\chi(\mathcal{U}) \leq 7$.

- The inequality $4 \leq \chi(\mathcal{U}) \leq 7$ was the best known evaluation of $\chi(\mathcal{U})$ for more than 60 years until quite recently.
  - The open problem asking the value of $\chi(\mathcal{U})$ is called Hadwiger-Nelson problem.
- In April 2018, an amateur mathematician Aubrey de Grey uploaded a preprint in which he gives a finite graph with more than 1000 points which is embeddable into $\mathcal{U}$ and of chromatic number 5. This improves the inequality above to

$$5 \leq \chi(\mathcal{U}) \leq 7$$

.

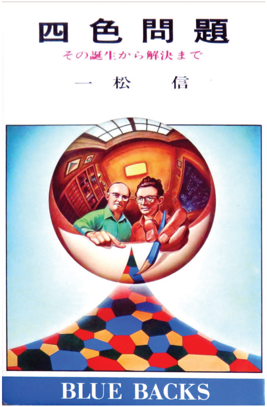
$5 \leq \chi(\mathcal{U}) \leq 7$

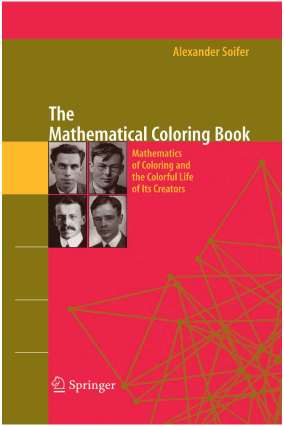
- ▷ De Grey used a computer program to calculate the chromatic number of some of the finite (unit distance planer) graphs. These graphs are chosen in the way that they are built up systematically from subgraphs which have only few number of good 4 coloring. This makes the checking of whether the whole graph is non 4 chromatic chromatic feasible by computer calculation. As far as I understand the search of a desired graph by de Grey was otherwise a matter of "try and error".

- ► In this regard, the construction does not give much new insight except that there is a finite graph showing that $5 \leq \chi(\mathcal{U})$.

- ▷ However, now that many mathematicians became aware of the problem through the discovery of the 5 chromatic graph, a chain reaction of new developments in connection with the Hadwiger-Nelson Problem might follow this breakthrough.

Recommendations for further reading



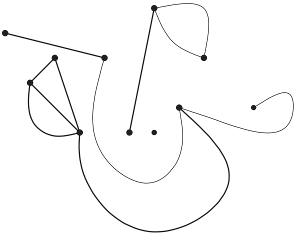


"The High Line" in Kobe
The former freight train line 神戸臨港線
near the Hyogo International House


Dziękuję za uwagę.
御清聴ありがとうございました。

Graphs

- A figure consisting of several points and lines connecting some pairs of the points is called a graph.



- Two points in a graph are said to be adjacent if they are connected directly by a line.
- A graph is connected if each two distinct points of the graph are connected by a chain of lines. The graph above is not connected. If a graph is not connected then it clearly cannot be drawn by a single stroke. Thus in the following we shall consider only connected graphs.

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Subgraphs of a graph

- If G is a graph, a graph H is called a subgraph of G if the points and lines of H consist of (some of the) points and lines of G.

- If H is a subgraph of G, then we have $\chi(H) \leq \chi(G)$.

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Proof of Theorem 1

Theorem 1 (Euler, 1736). Suppose that a graph G can be drawn in a single stroke. Then either (1) the degree of each point of the graph is an even number, or (2) there are exactly two points of the graph with odd degree.

Proof. **Case 1:** G can be drawn in a single stroke in such a way that the starting point of the stroke is different from the final point.

In this case, I claim that the starting point and the end point should be of odd degree while all other points should have even degree. That is, we have (2) in this case.

Case 2: G can be drawn in a single stroke in such a manner that the starting point of the stroke is identical with the end point of the stroke.

Similarly to Case 1, we prove in this case that all the points of the graph should have even degree. That is, we have (1) in this case. □

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Contraposition (対偶)

- For a statement of the form

  "if A holds then B holds",

  the statement of the form

  "if B does not hold then A does not hold"

  for the same assertions A and B is called the contrapositive of the original statement. (contrapositive: 対偶命題)

- Any statement and its contrapositive are logically equivalent.

- Caution: This is not applicable for sentences which are not expressing statements whose truth and falsity can be discussed.

  Example: "If it rains then I take an umbrella." versus "If I don't take an umbrella, then it does not rain."

Back

Converse (逆)

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- For a statement of the form

  "if A holds then B holds",

  the statement of the form

  "if B holds then A holds"

  is called the converse of the original statement. (converse: 逆命題)

- A statement and its converse are not necessarily logically equivalent.

  「逆は必ずしも真ならず.」

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