On the possible solution(s) of the Continuum Problem

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The Main Thesis

The Main Thesis. If one of the reasonable strong enough reflection principles should be assumed (as an additional set-theoretic axiom), then the continuum is either $\aleph_1$ or $\aleph_2$ or extremely large.

- The adjective "reasonable" in the statement above might be subjective.

- Still I am going to try to convince you in this talk that the whole statement has certain degree of objectivity.

Reflection Principles

- We consider the following type of statements:

$\mathsf{RP}_{\mathcal{C},<\kappa}$ For the class $\mathcal{C}$ of structures, if $\mathfrak{A} \in \mathcal{C}$ then there is a substructure $\mathfrak{B} \in \mathcal{C}$ of $\mathfrak{A}$ of cardinality $< \kappa$.

- Note that Downward Löwenheim-Skolem theorems (DLSTs) can be seen as statements of this type. For example, the usual DLST for first-order logic can be formulated as $\mathsf{RP}_{\mathcal{C}<\kappa}$ for

$$\mathcal{C} = \{\mathfrak{A} : \mathfrak{A} \text{ is an infinite structure in countable language with built-in Skolem-functions}\};$$

$$\kappa = \aleph_1.$$

- Let us call a statement of the form $\mathsf{RP}_{\mathcal{C},<\kappa}$ a reflection principle (RP, for short) and "$<\kappa$" the reflection point of the reflection principle.

Reflection Principles (2/2)

- Sometimes it is convenient to consider some refinement of the substructure relation of the elements of the class of structures in the Reflection Principles:

$\mathrm{RP}_{\mathcal{C},\sqsubseteq,<\kappa}$ For the class $\mathcal{C}$ of structures, and a binary relation $\sqsubseteq$ on $\mathcal{C}$ which refines the substructure relation on the elements of $\mathcal{C}$, if $\mathfrak{A} \in \mathcal{C}$ then there is a $\mathfrak{B} \in \mathcal{C}$ s.t. $\mathfrak{B} \sqsubseteq \mathfrak{A}$ of cardinality $< \kappa$.

- In this framework, the usual DLST can be more naturally formulated as $\mathrm{RP}_{\mathcal{C},\sqsubseteq,<\kappa}$ for

$$\mathcal{C} := \{\mathfrak{A} : \mathfrak{A} \text{ is an infinite structure in a countable language}\};$$

$$\sqsubseteq := \text{the elementary substructure relation } \prec \ ;$$

$$\kappa := \aleph_1.$$

Some more RPs provable in ZFC

Theorem 1. (Alan Dow, 1988) For any compact Hausdorff space X, if X is not metrizable, then X has a subspace of size $< \aleph_2$ which is not metrizable.

▷ For a proof, see [dow] or [fuchino] .

Theorem 2. (DLST for $L(Q)$) Let $L(Q)$ be the logic obtained from the first-order logic by adding the (unary) quantifier Q where $Qx\,(...)$ is interpreted as "there are uncountably many x s.t. ...". Then, for any uncountable structure $\mathfrak{A}$ (in a countable language), there is $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$ of cardinality $< \aleph_2$.

Proof.

The Continuum Hypothesis is a Reflection Principle

- Let $L^{\aleph_0,II}$ be the monadic second-order logic where the second order variable X, Y etc. run over countable subsets of the underlying set of the structure in question (suggested by "$\aleph_0$"). As usual, the logic has the built-in binary relation ε where, for a first order variable x and a second order X, "$x \,\varepsilon\, X$" is interpreted as "x is an element of X". The logic allows $\forall$ and $\exists$ quantification the second-order variables (suggested by "II").

- For structures $\mathfrak{A}$, $\mathfrak{B}$ of countable language with $\mathfrak{B} \subseteq \mathfrak{A}$, we say that $\mathfrak{B} = \langle B, ...\rangle$ is a weak $L^{\aleph_0,II}$-elementary substructure of $\mathfrak{A}$ (notation: $\mathfrak{B} \prec^{-}_{L^{\aleph_0,II}} \mathfrak{A}$) if, for any $L^{\aleph_0,II}$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ in the language of $\mathfrak{A}$ <u>without</u> <u>second-order</u> <u>free</u> <u>variables</u>, and $b_0, ..., b_{n-1} \in B$, we have

$$\mathfrak{B} \models \varphi(b_0, ..., b_{n-1}) \;\Leftrightarrow\; \mathfrak{A} \models \varphi(b_0, ..., b_{n-1}).$$

The Continuum Hypothesis is a Reflection Principle (2/2)

- If $\mathcal{C} = \{\mathfrak{A} : \mathfrak{A}$ is a structure in a countable language$\}$, we shall drop $\mathcal{C}$ from $\mathrm{RP}_{\mathcal{C},\sqsubseteq,<\kappa}$ and write $\mathrm{RP}_{\sqsubseteq,<\kappa}$.
  - Thus $\mathrm{RP}_{\prec^-_{L^{\aleph_0,II}},<\aleph_2}$ is the statement:

$(\mathrm{RP}_{\prec^-_{L^{\aleph_0,II}},<\aleph_2})$: For any structure $\mathfrak{A}$ in a countable language, there is a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $<\aleph_2$ s.t. $\mathfrak{B} \prec^-_{L^{\aleph_0,II}} \mathfrak{A}$.

Theorem 3. (S.F., A. Ottenbreit, and H. Sakai [I]) CH is equivalent to $\mathrm{RP}_{\prec^-_{L^{\aleph_0,II}},<\aleph_2}$.

Proof.

Stationary Logic

- Let $L^{\aleph_0}_{stat}$ be the be the monadic second-order logic where the second order variable X, Y etc. run again over countable subsets of the underlying set of the structure in question. The built-in predicate ε is just like in case of $L^{\aleph_0,II}$. $L^{\aleph_0}_{stat}$ does not allow the second-order quantification but has the new second-order quantifier "*stat* $X(\ldots)$" whose interpretation is "there are stationarily many X s.t. …". In the literature $L^{\aleph_0}_{stat}$ is often referred to as stationary logic.

- The elementarity $\prec^{-}_{L^{\aleph_0}_{stat}}$ is defined similarly as before.

- $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$ is thus the principle:

$\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$: For any structure $\mathfrak{A}$ in a countable language, there is a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $< \aleph_2$ s.t. $\mathfrak{B} \prec^{-}_{L^{\aleph_0}_{stat}} \mathfrak{A}$.

Stationary Logic (2/2)

$\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$: For any structure $\mathfrak{A}$ in a countable language, there is a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $< \aleph_2$ s.t. $\mathfrak{B} \prec^{-}_{L^{\aleph_0}_{stat}} \mathfrak{A}$.

- $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$ implies the principle called RP in [millennium-book] (Definition 37.17):

RP : For every regular $\lambda \geq \aleph_2$, if S is a stationary subset of $[\lambda]^{\aleph_0}$, then for any $X \in [\lambda]^{\aleph_1}$, there is $Y \in [\lambda]^{\aleph_1}$ s.t. $X \subseteq Y$ and $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$.

Proposition 4. (S.F., Ottenbreit and Sakai, [II]) $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$ implies RP.

Proof. Back to Cor.10.

Corollary 5. $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$ implies that $2^{\aleph_0} \leq \aleph_2$.

Proof. By Proposition 4. and by the fact that RP implies $2^{\aleph_0} \leq \aleph_2$ (Theorem 27.18 (Todorčević) ([millennium-book]). □ (Corollary 5.)

$2^{\aleph_0} = \aleph_2$ follows from a RP

- The following Proposition can be proved using Corollary 5. and Theorem 3.2 (a) by Baumgartner and Taylor in [baumgartner-taylor] :

Theorem 3.2

Proposition 6. (S.F., A. Ottenbreit, and H. Sakai, [II])
$RP_{\prec^-_{L^{\aleph_0}_{stat}}, <\kappa}$ for $\kappa > \aleph_2$ implies $\kappa > 2^{\aleph_0}$.

Corollary 7. (S.F., A. Ottenbreit, and H. Sakai, [II])
$RP_{\prec^-_{L^{\aleph_0}_{stat}}, <2^{\aleph_0}}$ implies $2^{\aleph_0} = \aleph_2$.

Proof.
- $RP_{\prec^-_{L^{\aleph_0}_{stat}}, <2^{\aleph_0}}$ implies $2^{\aleph_0} \leq \aleph_2$.
  [If $2^{\aleph_0} > \aleph_2$, then $2^{\aleph_0} > 2^{\aleph_0}$ by Proposition 6. This is a contradiction.]
- $RP_{\prec^-_{L^{\aleph_0}_{stat}}, <\aleph_1}$ does not hold.
  ["there exists uncountably many x s.t. ..." is expressible in $L^{\aleph_0}_{stat}$ (Lemma 4a.)]
- Thus, $2^{\aleph_0} \neq \aleph_1$ and hence $2^{\aleph_0} = \aleph_2$.

□ (Corollary 7.)

Game Reflection Principle

- For a set A and $\mathcal{A} \subseteq {}^{\omega_1>}A$, we define the game $\mathcal{G}(\mathcal{A})$ in which two players I and II choose elements of A alternately:

$$\begin{array}{c|cccccccccc} \text{I} & a_0 & & a_1 & & a_2 & & \cdots & a_\xi & & \cdots \\ \hline \text{II} & & b_0 & & b_1 & & b_2 & \cdots & & b_\xi & \cdots \end{array} \qquad (\xi < \omega_1)$$

- II wins the game, if
  - ▷ $\langle a_\xi, b_\xi \ : \ \xi < \eta \rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi \ : \ \xi < \eta \rangle ^\frown \langle a_\eta \rangle \notin \mathcal{A}$ for any $a_\eta \in A$ some $\eta < \omega_1$; or
  - ▷ $\langle a_\xi, b_\xi \ : \ \xi < \omega_1 \rangle \in [\mathcal{A}]$

  where $[\mathcal{A}] := \{ f \in {}^{\omega_1}A \ : \ f \restriction \nu \in \mathcal{A} \text{ for all } \nu < \omega_1 \}$.

- The Game Reflection Principle (GRP) [könig] (Strong Game Reflection Principle in B. König's terminology) is the following principle:

GRP : For any set A of regular cardinality, $\mathcal{A} \subseteq {}^{\omega_1>}A$, and for ω_1-club $\mathcal{C} \subseteq [A]^{\aleph_1}$, if the player II does not have a winning strategy in $\mathcal{G}(\mathcal{A})$ then there is a $B \in \mathcal{C}$ s.t. II does not have a winning strategy in $\mathcal{G}(\mathcal{A} \cap {}^{\omega_1>}B)$.

Game Reflection Principle (2/3)

GRP : For any set A of regular cardinality, $\mathcal{A} \subseteq {}^{\omega_1>}A$, and for ω_1-club $\mathcal{C} \subseteq [A]^{\aleph_1}$, if the player *II* does not have a winning strategy in $\mathcal{G}(\mathcal{A})$ then there is a $B \in \mathcal{C}$ s.t. *II* does not have a winning strategy in $\mathcal{G}(\mathcal{A} \cap {}^{\omega_1>}B)$.

- GRP is also a principle of the type $\mathrm{RP}_{\mathcal{C},\sqsubseteq,<\aleph_2}$.
- This follows among other things from the following:

Theorem 8. (B. König [könig]) GRP implies CH.

Theorem 9. (S.F., A. Ottenbreit, and H. Sakai, [I]) GRP implies $\mathrm{RP}_{\prec^-_{L^{\aleph_0}_{stat}},<\aleph_2}$.

Corollary 10. GRP implies the Singular Cardinal Hypothesis (SCH).

Game Reflection Principle (3/3)

Corollary 10. GRP implies the Singular Cardinal Hypothesis (SCH).

see the proof of Proposition 5. + by definition

Proof. GRP $\underbrace{\Rightarrow}_{\text{Theorem 9}}$ $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0}_{stat}}, <\aleph_2}$ $\underbrace{\Rightarrow}_{\text{Proposition 4.}}$ RP $\overbrace{\Rightarrow}$ $2^{\aleph_0} \leq \aleph_2$ + FRP

$\underbrace{\Rightarrow}_{\text{[fuchino-rinot]}}$ SCH

□ (Corollary 10.)

Theorem 11. ([könig]) GRP is equivalent to the assertion: "$\aleph_2$ is generically supercompact by σ-closed p.o.s".

(Existence of) generic large cardinals as Reflection Principles

Theorem 11. ([könig]) GRP is equivalent to the assertion:
"$\aleph_2$ is generically supercompact by σ-closed p.o.s".

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is generically supercompact by $\mathcal{P}$, if for any $\lambda \geq \kappa$ there is a $\mathbb{P} \in \mathcal{P}$ s.t. for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are classes M, $j \in \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

▷ $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$:
j is an elementary embedding of V to M;
M is a transitive class;
$j(\alpha) = \alpha$ for all $\alpha < \kappa$;
and $j(\kappa) > \kappa$
(κ is a critical point of j).
This is formalizable in ZFC !!
(see [the higher inf.] 5.1 Proposition)

The closedness condition $j''\lambda \in M$

- The supercompactness of a cardinal κ is defined by the existence of j, $M \subseteq V$ for any $\lambda \geq \kappa$ s.t. $j : V \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \lambda$, and $[M]^{\lambda} \subseteq M$.
  - The last condition (the closedness of M) is too strong for a "generic" version of the supercompactness, in general. The condition "$j''\lambda \in M$" is a replacement of this closedness of M.

Lemma 12. (Lemma 2.5 in [I]) Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter for a p.o. $\mathbb{P} \in \mathsf{V}$, and $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M \subseteq \mathsf{V}[\mathbb{G}]$ with $j''\lambda \in M$ for a $\lambda \geq \kappa$. Then, we have the following:

(1) For any set $A \in \mathsf{V}$ with $\mathsf{V} \models |A| \leq \lambda$, we have $j''A \in M$.

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$.

(3) For any $A \in \mathsf{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.

(4) $(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, Thus, if $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$. (6) $j \restriction A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.

Lévy Collapse

- For a set $S \subseteq \mathrm{On}$ and an infinite regular cardinal λ, let

  $\mathrm{Col}(\lambda, S) := \{f :$
  f is a mapping with $\mathrm{dom}(f) \subseteq (S \setminus 2) \times \lambda$, $\mathrm{rng}(f) \subseteq \sup S$,
  $|f| < \lambda$, for all $\langle\alpha,\xi\rangle \in \mathrm{dom}(f)\,(f(\langle\alpha,\xi\rangle < \alpha))\}$,

  $\mathbb{1}_{\mathrm{Col}(\lambda,S)} := \emptyset$, and

  $f \leq_{\mathrm{Col}(\lambda,S)} g \;:\Leftrightarrow\; g \subseteq f$ for $f, g \in \mathrm{Col}(\lambda, S)$.

- $\mathrm{Col}(\lambda, S)$ adds surjections from λ to α for each $\alpha \in S$.

Lemma 13. (see e.g. 10.17 Lemma in [the higher inf.])

(1) Suppose κ, μ are infinite regular cardinal with $\mu < \kappa$. If κ is an inaccessible cardinal, then $\mathrm{Col}(\mu, \kappa)$ has the κ-cc.

(2) Suppose κ, μ are infinite regular cardinal with $\mu < \kappa$. If κ is an inaccessible cardinal or $\mu = \omega$, then $\mathrm{Col}(\mu, \kappa)$ forces that all ordinals α with $\mu \leq \alpha < \kappa$ to be of cardinality μ and preserves all cardinals and cofinality $\geq \kappa$.

(3) If $S = X \dot\cup Y$ then $\mathrm{Col}(\mu, S) \cong \mathrm{Col}(\mu, X) \times \mathrm{Col}(\mu, Y)$.

A model of "$\aleph_2$ is generically supercompact ..."

- By the following theorem with $\mu = \aleph_1$, we obtain a model in which $\aleph_2$ is generically supercompact by σ-closed p.o.s.

Theorem 14. Suppose that κ is a supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \mathrm{Col}(\mu, \kappa)$. Then, for a $(\mathsf{V}, \mathbb{P}_0)$-generic $\mathbb{G}_0$,

▷ $\mathsf{V}[\mathbb{G}_0] \models$" μ^+ is generically supercompact by $<\mu$-closed p.o.s".

Proof. Note that $\mathsf{V}[\mathbb{G}_0] \models$" $\mu^+ = \kappa$".

For $\lambda \geq \kappa$, let $j : \mathsf{V} \xrightarrow{\preceq} M$ be a λ-supercompact embedding for κ.
Then we have $j(\mathbb{P}_0) \underbrace{=}_{\text{by elementarity}} \mathrm{Col}(\underbrace{j(\mu)}_{=\mu}, j(\kappa))^M \underbrace{=}_{\text{by closedness of } M} \mathrm{Col}(\mu, j(\kappa))^{\mathsf{V}}$.

For a $(\mathsf{V}[\mathbb{G}_0], \mathrm{Col}(\mu, j(\kappa) \setminus \kappa))$-generic filter $\mathbb{G}$, the lifting

$\tilde{j} : \mathsf{V}[\mathbb{G}_0] \xrightarrow{\preceq} \underbrace{M[\mathbb{G}_0][\mathbb{G}]}_{\subseteq \mathsf{V}[\mathbb{G}_0][\mathbb{G}]}$; $\quad \underset{\sim}{a}^{\mathbb{G}_0} \mapsto j(\underset{\sim}{a})^{\mathbb{G}_0 * \mathbb{G}}$

witnesses the generic λ-supercompactness of $\overbrace{\kappa}^{=(\mu^+)^{\mathsf{V}[\mathbb{G}_0]}}$ by μ-closed p.o.s in $\mathsf{V}[\mathbb{G}_0]$.

□ (Theorem 14.)

Laver-generic supercompact cardinals

- The proof of Theorem 14. can be yet refined to obtain the following:

Theorem 15. Suppose that κ is a supercompact cardinal, and $\mathbb{P}_0 = \mathrm{Col}(\aleph_1, \kappa)$. Then, for a $(\mathsf{V}, \mathbb{P}_0)$-generic $\mathbb{G}_0$,

▷ $\mathsf{V}[\mathbb{G}_0] \models$ "$\aleph_2$ is Laver-generically supercompact for σ-closed p.o.s".

Here, a cardinal κ is said to be Laver-generically supercompact for a class $\mathcal{P}$ of p.o.s, if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name of a p.o. $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \mathrel{\varepsilon} \mathcal{P}$" s.t., for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$, there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ s.t.

▷ $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$,

▷ $j(\kappa) > \lambda$, ▷ $\mathbb{P}, \mathbb{H} \in M$ and ▷ $j''\lambda \in M$.

- The definition of Laver-generic supercompactness is slightly stronger than the one given in [II].

The Trichotomy

- If $\mathcal{P}$ in the definition of Laver-generic large cardinal is taken to be some natural class of p.o.s then we obtain the trichotomy mentioned at the beginning of the talk. In particular:

Theorem 16. ([II]) (1) Suppose that μ is Laver-generically supercompact for σ-closed p.o.s. Then, $2^{\aleph_0} = \aleph_1$, $\mu = \aleph_2$, and $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ holds.

(2) Suppose that μ is Laver-generically supercompact for proper p.o.s. Then $2^{\aleph_0} = \mu = \aleph_2$, and $\mathsf{PFA}^{+\omega_1}$ holds.

(3) Suppose that μ is Laver-generically superhuge for ccc p.o.s. Then $2^{\aleph_0} = \mu$ and $\mathcal{P}_\mu(\lambda)$ for any regular $\lambda \geq \mu$ carries an $\aleph_1$-saturated normal ideal. In particular, μ is μ-weakly Mahlo. Also $\mathsf{MA}^{++\kappa}(ccc, < \mu)$ for all $\kappa < \mu$ holds.

Lever-generic large cardinals are definable

- We called this notion of generic large cardinals "Laver-generic large cardinal" since we need to iterate large cardinal times along with a "Laver diamond" to obtain models for (2) and (3) of Theorem 16.

- "Laver-generic large cardinals" are first order definable. This is not at all trivial and is proved in [fuchino-sakai] .

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ご清聴ありがとうございました.
Thank you for your attention!

Definition of some notations

- For an ordinal α and a set A, ${}^{\alpha}A := \{f : f : \alpha \to A\}$.
  - For an ordinal β, ${}^{\beta>}A := \bigcup_{\alpha<\beta} {}^{\alpha}A$. Back

- For a set A and a cardinal κ, $[A]^{\kappa} := \{a \in \mathcal{P}(A) : |a| = \kappa\}$.

- $\mathcal{C} \subseteq [A]^{\aleph_1}$ is ω_1-club if
  - for all $a \in [A]^{\aleph_1}$ there is $b \in \mathcal{C}$ with $a \subseteq b$ (unbounded or cofinal); and
  - for any $\subseteq$-increasing sequence $\langle a_\xi : \xi < \omega_1\rangle$ of elements of $\mathcal{C}$, we have $\bigcup_{\xi<\omega_1} a_\xi \in \mathcal{C}$. Back

Stationarity of sets of countable sets

- For a set X, we write
  $[X]^{\aleph_0} := \{a : a \subseteq X, \text{ and } a \text{ is countable}\}$.

- $C \subseteq [X]^{\aleph_0}$ is closed unbounded (club), if
  - For any $a \in [X]^{\aleph_0}$ there is $b \in C$ s.t. $a \subseteq b$ (unbounded or cofinal); and
  - For any increasing sequence $\langle a_n : n \in \omega \rangle$ in $\mathcal{C}$, (i.e. $a_n \in \mathcal{C}$ for all $n \in \omega$ and, for $n, n' \in \omega$ with $n < n'$, $a_n \subseteq a_{n'}$), we have $\bigcup_{n\in\omega} a_n \in C$. (closed)

- $S \subseteq [X]^{\aleph_0}$ is stationary, if $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^{\aleph_0}$.

Back

Proof of Proposition 4.

RP : For every regular $\lambda > \aleph_2$, if S is a stationary subset of $[\lambda]^{\aleph_0}$, then for any $X \in [\lambda]^{\aleph_1}$, there is $Y \in [\lambda]^{\aleph_1}$ s.t. $X \subseteq Y$ and $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$.

Proposition 4. (S.F., Ottenbreit and Sakai, [II]) $\mathrm{RP}_{\prec^-_{L^{\aleph_0}_{stat}}, <\aleph_2}$ implies RP.

Lemma 4a. "there exist uncountably many x s.t. ..." is expressible in $L^{\aleph_0}_{stat}$.

Proof. "$stat\, X \exists x\, (x \not\in X \wedge \ldots)$" will do. □ (Lemma 4a)

Proof of Proposition 4. Suppose that $S \subseteq [\lambda]^{\aleph_0}$ is stationary and $X \subseteq [\lambda]^{\aleph_1}$.

- Let κ be a sufficiently large regular cardinal and

$$\mathfrak{A} := \langle \mathcal{H}(\kappa), \lambda, S, X, \in \rangle$$

- Let $\mathfrak{B} = \langle B, \ldots \rangle$ be s.t. $\mathfrak{B} \prec^-_{L^{\aleph_0}_{stat}} \mathfrak{A}$ and $|B| < \aleph_2$.
- Then $Y := \lambda \cap B$ is as desired. □ (Propostion 4.)

Back

Proof of Theorem 3.

Theorem 3. (S.F., A. Ottenbreit, and H. Sakai [I]) CH is equivalent to $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0,II}}, <\aleph_2}$.

Proof. ▶ ($\Leftarrow$) : Assume that $\mathrm{RP}_{\prec^{-}_{L^{\aleph_0,II}}, <\aleph_2}$ holds and consider the structure

$$\mathfrak{A} := \langle \mathcal{P}(\omega), \overbrace{n}^{\text{constant}}, \underbrace{\omega}_{\text{unary relation}}, \overbrace{\in}^{\text{binary relation}} \rangle_{n\in\omega}.$$

▷ Note the formula $\forall X\,(\text{“}X \subseteq \underline{\omega}\text{”} \rightarrow \exists x \forall y\,(y \,\underline{\in}\, x \leftrightarrow y \,\varepsilon\, X))$.

▶ For every $\mathfrak{B} \prec^{-}_{L^{\aleph_0,II}} \mathfrak{A}$, we have $\mathfrak{B} = \mathfrak{A}$. Thus CH holds.

($\Rightarrow$) : ▶ Assume that CH holds and let $\mathfrak{A} = \langle A, \ldots \rangle$ be a structure in a countable language.

▶ Let κ be a regular cardinal s.t. $\mathfrak{A} \in \mathcal{H}(\kappa)$.

▷ By CH, there is $M \prec \mathcal{H}(\kappa)$ s.t. $\mathfrak{A} \in M$, $|M| = \aleph_1$ and $[M]^{\aleph_0} \subseteq M$.

▶ Let $\mathfrak{B} := \mathfrak{A} \restriction A \cap M$. Then $\|\mathfrak{B}\| \leq \aleph_1$ and $\mathfrak{B} \prec^{-}_{L^{\aleph_0,II}} \mathfrak{A}$.

Proof of Theorem 2.

Theorem 2. (DLST for $L(Q)$) Let $L(Q)$ be the logic obtained from the first-order logic by adding the (unary) quantifier Q where $Qx\,(...)$ is interpreted as "there are uncountably many x s.t. ...". Then, for any uncountable structure $\mathfrak{A}$ (in a countable language), there is $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$ of cardinality $< \aleph_2$.

Proof. Suppose that $\mathfrak{A} = \langle A, ...\rangle$ is a structure in a countable language.

$\mathcal{H}(\kappa) = \{x : |\,trcl(x)\,| < \kappa\}$

- Let κ be a regular cardinal with $\mathfrak{A} \in \mathcal{H}(\kappa)$. Let $M \prec \mathcal{H}(\kappa)$ be s.t. $\mathfrak{A} \in M$, $\omega_1 \subseteq M$, and $|\,M\,| = \aleph_1$.
- Let $B := A \cap M$ and $\mathfrak{B} := \mathfrak{A} \restriction B$. Then $\mathfrak{B}$ is of cardinality $< \aleph_2$ and $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$. □ (Theorem 2.) Back