Laver-generically large cardinal and the Continuum Problem

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Generically large cardinals

- For a p.o. $\mathbb{P}$, A cardinal μ is generically measurable by $\mathbb{P}$, if, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t.

(1) $j : V \overset{\preccurlyeq}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$; and (2) $crit(j) = \mu$.

- For a calss $\mathcal{P}$ of p.o.s, μ is generically measurable by $\mathcal{P}$, if μ is generically measurable by some $\mathbb{P} \in \mathcal{P}$.

Lemma 1. (1) If κ is measurable then κ is generically measurable by any class $\mathcal{P}$ of p.o.s with $\{\mathbb{1}\} \in \mathcal{P}$.

(2) Suppose that κ is measurable, $\aleph_0 < \delta < \kappa$ regular, and $\mathbb{P} = \mathrm{Col}(\delta, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, δ^+ $(= \kappa)$ is generically measurable by σ-closed p.o.s. In the generic extension, $2^{\aleph_0}$ can be anything of uncountable cofinality between $\aleph_1$ and δ.

(3) Suppose that κ is measurable, and $\mathbb{P}$ is a p.o. for adding $\geq \kappa$ Cohen reals. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\kappa \leq 2^{\aleph_0}$ and κ is generically measurable by p.o.s adding Cohen reals.

Proof.

Generically large cardinals (2/4)

Lemma 2. Suppose that μ is a generically measurable cardinal by some $\mathbb{P}$. Then, (1) μ is regular.

(2) If $\mathbb{P}$ is $\aleph_1$ preserving then $\mu > \aleph_1$.

Proof. (1): Suppose not and let $f : \mu_0 \to \mu$ be cofinal with $\mu_0 < \mu$.

▷ Let $\mathbb{G}$, j, M be as in the definition of generic measurability by $\mathbb{P}$. Then $j(f) = f$ by elementarity and $crit(j) = \mu$.

▷ By elementarity,

$$M \models j(\mu) = \sup(\underbrace{j(f)}_{=f}) = \mu$$

▷ This is a contradiction to $\mu = crit(j)$.

▶ (2): Suppose not. Then $\mu = \omega_1$.

▷ Let $\mathbb{G}$, j, M be as in the definition of generic measurability by $\mathbb{P}$. Then $M \models$"$j(\mu) = \omega_1$". Hence $M \models$"μ is countable". Thus $V[\mathbb{G}] \models$"μ is countable".

▷ This is a contradiction to the assumption on $\mathbb{P}$. □ (Lemma 2)

Back to the proof of Theorem 6, (3). Back to the proof of Theorem 8.

Generically large cardinals (3/4)

- For a class of p.o.s $\mathcal{P}$, a cardinal μ is generically supercompacrt (generically super-almosthuge or generically superhuge, resp.) by $\mathcal{P}$ if, for any $\lambda \geq \mu$, there are $\mathbb{P} \in \mathcal{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t.

(1) $j : V \overset{\preceq}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$,

(2) $crit(j) = \mu$, $j(\mu) > \lambda$,

(3) $j''\lambda \in M$ $\quad$ ($j''\delta \in M$ for all $\delta < j(\mu)$ or $j''j(\mu) \in M$, resp.)

Generically large cardinals (4/4)

- The following Lemma is similar to Lemma 1:

Lemma 3. (1) If κ is supercompact (super-almosthuge, or superhuge, resp.) then κ is generically supercompact (super-almosthuge, or superhuge, resp.) by $\mathcal{P} = \{\mathbb{P}\}$ for $\mathbb{P} = \{\mathbb{1}\}$.

(2) Suppose that κ is supercompact (super-almosthuge, or superhuge, resp.), $\aleph_0 < \delta < \kappa$ regular, and $\mathbb{P} = \mathrm{Col}(\delta, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, δ^+ $(= \kappa)$ is generically supercompact (super-almosthuge, or superhuge, resp.) by σ-closed p.o.s. In the generic extension, $2^{\aleph_0}$ can be anything of uncountable cofinality between $\aleph_1$ and δ.

(3) Suppose that κ is supercompact (super-almosthuge, or superhuge, resp.), and $\mathbb{P}$ is a p.o. for adding $\geq \kappa$ Cohen reals. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\kappa \leq 2^{\aleph_0}$ and κ is generically supercompact (super-almosthuge, or superhuge, resp.) by p.o.s adding Cohen reals. $\square$

"$j''\lambda \in M$" as a closure property

- "$j''\lambda \in M$" in the definition of generic large cardinals is a closure property of M:

Lemma 4 (Folklore, [II]). Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter for a p.o. $\mathbb{P} \in \mathsf{V}$ and $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}[\mathbb{G}]$ s.t., for cardinals κ, λ in V with $\kappa \leq \lambda$, $crit(j) = \kappa$ and $j''\lambda \in M$.

(1) For any set $A \in \mathsf{V}$ with $\mathsf{V} \models |A| \leq \lambda$, we have $j''A \in M$.

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$.

(3) For any $A \in \mathsf{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.

(4) $(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, Thus, if $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.

(6) $j \restriction A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.

[II] S.F., André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics II — reflection down to the continuum, to appear in Archive for Mathematical Logic (2021).

Back to the proof of Theorem 8.

Laver-generically large cardinal

- A class $\mathcal{P}$ of p.o.s is iterable if $\mathbb{P} \in \mathcal{P}$ and $\Vdash_{\mathcal{P}}$"$\underset{\sim}{\mathbb{R}} \in \mathcal{P}$" then $\mathbb{P} * \underset{\sim}{\mathbb{R}} \in \mathcal{P}$.
- For an iterable class of p.o.s $\mathcal{P}$, a cardinal μ is Laver-generically supercompacrt (Laver-generically super-almosthuge or Laver-generically superhuge, resp.) for $\mathcal{P}$ if, for any $\lambda \geq \mu$, and $\mathbb{P} \in \mathcal{P}$, there are $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t.

$(0)^*$ $\mathbb{Q} \cong \mathbb{P} * \underset{\sim}{\mathbb{R}}$ for a $\mathbb{P}$-name $\underset{\sim}{\mathbb{R}}$ with $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{R}} \in \mathcal{P}$",

(1) $j : V \xrightarrow{\preccurlyeq} M \subseteq \mathsf{V}[\mathbb{H}]$,

(2) $crit(j) = \mu$, $j(\mu) > \lambda$,

$(2\,¼)^*$ $\mathbb{P}$, $\mathbb{H} \in M$,

$(2\,½)^*$ $|\,\mathbb{Q}\,| \leq j(\mu)$,

(3) $j''\lambda \in M$ ($j''\delta \in M$ for all $\delta < j(\mu)$ or $j''j(\mu) \in M$, resp.)

Consistency of Laver-generically large cardinals

Lemma 5. (1) Suppose that κ is supercompact (super-almosthuge, or superhuge, resp.) and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2$ $(= \kappa)$ is Laver-generically supercompact (super-almosthuge, or superhuge, resp.) for σ-closed p.o.s.

(2) Suppose that κ is super-almosthuge (or superhuge, resp.) with a Laver function f, and $\mathbb{P}$ is the CS-iteration for forcing PFA along f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2$ $(= 2^{\aleph_0} = \kappa)$ is Laver-generically super-almosthuge (or superhuge, resp.) for proper p.o.s.

(3) Suppose that κ is supercompact (super-almosthuge, or superhuge, resp.) and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is Laver-generically supercompact (super-almosthuge, or superhuge, resp.) for Cohen p.o.s.

(4) Suppose that κ is supercompact (super-almosthuge, or superhuge, resp.) with a Laver function f, and $\mathbb{P}$ is a FS-iteration for forcing MA along f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is Laver-generically supercompact (super-almosthuge, or superhuge, resp.) for c.c.c. p.o.s. □

The continuum under a Laver-generically large cardinal

Proposition 6 ([II]). (1) Suppose that μ is Laver-genenerically supercompact for an iterable class $\mathcal{P}$ of ω_1-preserving p.o.s s.t. there is a $\mathbb{P}^* \in \mathcal{P}$ which collapses ω_2. Then $\mu = \omega_2$. Proof.

(2) Suppose that μ is Laver-generically supercompact for an iterable class $\mathcal{P}$ of p.o.s with at least one $\mathbb{P}^* \in \mathcal{P}$ which adds a new real. Then $\mu \leq 2^{\aleph_0}$. Proof.

(3) Suppose that μ is generically supercompact by a class $\mathcal{P}$ of p.o.s s.t. no $\mathbb{P} \in \mathcal{P}$ adds any real(∗). Then $2^{\aleph_0} < \mu$. Proof.

(∗) Here, the generic supercompactness (without "Laver") is enough.

[II] S.F., André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics II — reflection down to the continuum, to appear in Archive for Mathematical Logic (2021).

Back to the proof of Theorem 8.

Theorem 7 (Proposition 2.8 in [II]). Suppose that μ is Laver-generically supercompact for c.c.c. p.o.s. Then,

(1) SCH holds.

(2) there is a σ-saturated normal filter over $\mathcal{P}_\mu(\lambda)$ for all regular $\lambda \geq \mu$. □

Theorem 8 (Theorem 5.8 in [II]). Suppose that μ is Laver-generically superhuge for c.c.c. p.o.s. Then $\mu = 2^{\aleph_0}$.

Proof.

Problem. Does Theorem 8 hold for Laver-generic supercompactness?

[II] S.F., André Ottenbreit Maschio Rodrigues and Hiroshi Sakai,
Strong downward Löwenheim-Skolem theorems for stationary logics II
— reflection down to the continuum,
to appear in Archive for Mathematical Logic (2021).

MA^{++} under Laver-generically large cardinal

Theorem 9 (Theorem 5.7 in [II]). (1) For a class $\mathcal{P}$ of ccc p.o.s, if μ is Laver-generically supercompact for $\mathcal{P}$, then $\mathsf{MA}^{++\kappa}(\mathcal{P}, <\mu)$ holds for all $\kappa < \mu$.

(2) If $\aleph_2$ is Laver-generically supercompact for an iterable class $\mathcal{P}$ of p.o.s which preserves stationarity of subsets of ω_1, then $\mathsf{MA}^{+\omega_1}(\mathcal{P})$ holds.

$\mathsf{MA}^{++\kappa}(\mathcal{P}, <\mu)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mu$ and any family $\mathcal{S}$ of $\mathbb{P}$-names s.t. $|\mathcal{S}| \leq \kappa$ and $\Vdash_{\mathbb{P}}$" $\utilde{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\utilde{S}}}(\theta_{\utilde{S}})$" for some $\omega < \eta_{\utilde{S}} \leq \theta_{\utilde{S}} \leq \kappa$ with $\eta_{\utilde{S}}$ regular, for all $\utilde{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ s.t. $\utilde{S}(\mathbb{G})$ is stationary in $\mathcal{P}_{\eta_{\utilde{S}}}(\theta_{\utilde{S}})$ for all $\utilde{S} \in \mathcal{S}$.

Summary

- By putting together the results explained so far, the following three scenarios stand out:

Conclusion 10 ([II]). (1) Suppose that μ is Laver-generically supercompact for σ-closed p.o.s. Then, $2^{\aleph_0} = \aleph_1$, $\mu = \aleph_2$, and $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ holds.

(2) Suppose that μ is Laver-generically supercompact for proper p.o.s. Then $2^{\aleph_0} = \mu = \aleph_2$, and $\mathsf{PFA}^{+\omega_1}$ holds.

(3) Suppose that μ is Laver-generically superhuge for ccc p.o.s. Then $2^{\aleph_0} = \mu$ and $\mathcal{P}_\mu(\lambda)$ for any regular $\lambda \geq \mu$ carries an $\aleph_1$-saturated normal ideal. In particular, μ is μ-weakly Mahlo. Also $\mathsf{MA}^{++\kappa}(ccc, {<}\mu)$ for all $\kappa < \mu$ holds.

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Thank you for your attention!
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Proof of Lemma 1, (2)

- Lemma 1, (2) and (3) can be proved similarly.

(2) Suppose that κ is measurable, $\aleph_0 < \delta < \kappa$ regular, and $\mathbb{P} = \mathrm{Col}(\delta, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, δ^+ $(= \kappa)$ is generically measurable by σ-closed p.o.s. In the generic extension, $2^{\aleph_0}$ can be anything of uncountable cofinality between $\aleph_1$ and δ.

Proof. ▶ Let $\mathbb{P}$ and $\mathbb{G}$ be as above. Let $j : V \overset{\preceq}{\to} M$ be the elementary embedding characterizing the measurability with ${}^{\kappa}M \subseteq M$.

▷ Let $\mathbb{P}^* = j(\mathbb{P})$. Then, by elementarity and the closure property, $j(\mathbb{P}) = \mathrm{Col}(\delta, j(\kappa))$. Let $\mathbb{P}^* = j(\mathbb{P})$. Then we have $\mathbb{P}^* \sim \mathbb{P} \times \mathbb{P}^*$.

▷ Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}], \mathbb{P}^*)$-generic filter. Let $\mathbb{H}^*$ be the $(\mathsf{V}, \mathbb{P}^*)$-generic filter corresponding to $\mathbb{G} \times \mathbb{H}$.

- Then $j^* : \mathsf{V}[\mathbb{G}] \overset{\preceq}{\to} M[\mathbb{H}^*] \subseteq \mathsf{V}[\mathbb{G}][\mathbb{H}]$; $\utilde{a}^{\mathbb{G}} \mapsto j(\utilde{a})^{\mathbb{H}^*}$ witnesses the generic measurability of κ in $\mathsf{V}[\mathbb{G}]$. $\square$ (Lemma 1.)

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Proof of Theorem 8.

Theorem 8 (Theorem 5.8 in [II]). Suppose that μ is Laver-generically superhuge for c.c.c. p.o.s. Then $\mu = 2^{\aleph_0}$.

Proof. ▶ $\mu \leq 2^{\aleph_0}$ follows from Proposition 6, **(2)** .

▶ To prove $2^{\aleph_0} \leq \mu$, let $\lambda \geq \mu$, $2^{\aleph_0}$ be large enough and let $\mathbb{Q}$ be a ccc p.o. s.t. there are $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}[\mathbb{H}]$ with $crit(j) = \mu$, $\lambda < j(\mu)$, $|\mathbb{Q}| \leq j(\mu)$, $\mathbb{H} \in M$ and $j''j(\mu) \in M$.

▷ Since $M \models$"$j(\mu)$ is regular" (by Lemma 2, **(1)** and elementarity), $j(\mu)$ is regular in V (by Lemma 4, **(3)**).

▷ Thus, we have $\mathsf{V} \models$"$j(\mu)^{\aleph_0} = j(\mu)$" by SCH (Theorem 7, **(1)**).

▷ Since $\mathbb{Q}$ has the ccc and $|\mathbb{Q}| \leq j(\mu)$, it follows that $\mathsf{V}[\mathbb{G}] \models$"$2^{\aleph_0} \leq j(\mu)$". By Lemma 4, **(4)** , $(j(\mu)^+)^M = (j(\mu)^+)^{\mathsf{V}[\mathbb{G}]}$. Thus $M \models$"$2^{\aleph_0} \leq j(\mu)$".

▷ By elementarity, $\mathsf{V} \models$"$2^{\aleph_0} \leq \mu$". □ (Theorem 8.)

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Proof of Proposition 6, (3)

Proposition 6, (3) Suppose that μ is generically supercompact by a class $\mathcal{P}$ of p.o.s s.t. no $\mathbb{P} \in \mathcal{P}$ adds any real. Then $2^{\aleph_0} < \mu$.

Proof. Suppose that $\mu \leq 2^{\aleph_0}$. Let $\lambda > 2^{\aleph_0}, \mu$.

▷ Let $\mathbb{P} \in \mathcal{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ be s.t. $j : \mathsf{V} \xrightarrow{\preceq} M$, $crit(j) = \mu$, and $j(\mu) > \lambda$.

▷ Since $\mathsf{V} \models 2^{\aleph_0} \geq \mu$ by assumption, we have

$$M \models \text{``}\,|\underbrace{({}^{\omega}2)^{M}}_{\subseteq ({}^{\omega}2)^{\mathsf{V}[\mathbb{G}]} = ({}^{\omega}2)^{\mathsf{V}}}| \geq \underbrace{j(\mu)}_{> \lambda > (2^{\aleph_0})^{\mathsf{V}}}\text{''}.$$

This is a contradiction. □ (Proposition 6, (3))

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Proof of Proposition 6, (2)

Proposition 6, (2) Suppose that μ is Laver-generically supercompact for an iterable class $\mathcal{P}$ of p.o.s with at least one $\mathbb{P}^* \in \mathcal{P}$ which adds a new real. Then $\mu \leq 2^{\aleph_0}$.

Proof. Suppose that $\kappa < \mu$ and $\langle a_\alpha : \alpha < \kappa \rangle$ is a sequence of reals. It is enough to show that $\langle a_\alpha : \alpha < \kappa \rangle$ does not enumerate reals.

- ▷ Let $\mathbb{Q} \in \mathcal{P}$, $\mathbb{P}^* \lessdot \mathbb{Q}$ be with a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $\mathbb{P}^*, \mathbb{H} \in M$, $j : \mathsf{V} \overset{\prec}{\to} M$ and $crit(j) = \mu$.
- ▷ $j(\langle a_\alpha : \alpha < \kappa \rangle) = \langle a_\alpha : \alpha < \kappa \rangle$.
- ▷ Since M contains a new real coded by $\mathbb{P}^*$ part of $\mathbb{H}$, we have $M \models$" $\langle a_\alpha : \alpha < \kappa \rangle$ does not enumerate $2^{\aleph_0}$".
- ▷ By elementarity, $\mathsf{V} \models$" $\langle a_\alpha : \alpha < \kappa \rangle$ does not enumerate $2^{\aleph_0}$".

□ (Proposition 6, (2))

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Proof of Proposition 6, (1)

Proposition 6, (1) Suppose that μ is Laver-genenerically supercompact for an iterable class $\mathcal{P}$ of ω_1-preserving p.o.s s.t. there is a $\mathbb{P}^* \in \mathcal{P}$ which collapses ω_2. Then $\mu = \omega_2$.

Proof. We have $\mu \geq \omega_1$ by Lemma 2, (2) . Suppose $\mu > \aleph_2$.

- ▷ Suppose that $\mathbb{Q} \in \mathcal{P}$ be s.t. $\mathbb{P}^* \lessdot \mathbb{Q}$ with $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \xrightarrow{\preccurlyeq} M$, $crit(j) = \mu$ and $\mathbb{P}^*$, $\mathbb{H} \in M$.
- ▷ By elementarity, we have $M \models$“ $\underbrace{j(\omega_2^{\mathsf{V}})}_{=\omega_2^{\mathsf{V}}}$ is “ω_2” ”.
- ▷ On the other hand, $\mathbb{P}^*$ part of $\mathbb{H}$ is in M and it collapses ω_2 to be an ordinal of cardinality $\aleph_1$. This is a contradiction. $\square$ (Proposition 6, (1))

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