Set-theoretic aspects of topological reflection theorems

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Coauthors

- The following results are obtained in joint researches mainly with:

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  - **Hiroshi Sakai** (酒井拓史, 神戸大学)
  - **Lajos Soukup** (Hungarian Academy of Science, Budapest, Hungary)
  - **Toshimichi Usuba** (薄葉季路，名古屋大学)

[1] S.F., **István Juhász**, Lajos Soukup, **Zoltán Szentmiklóssy** and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (June 2010), Special Issue dedicated to the Proceedings of the Conference "Advances in Set-Theoretic Topology" (in Honour of Tsugunori Nogura on his 60th Birthday), 1415-1429.

[2] S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba, More about Fodor-type Reflection Principle, preprint.

[3] S.F., Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, to appear in Fund. Math.

Reflection theorems on metrizability

Theorem 1 (A. Dow, 1988)

For a countably compact topological space X,

if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

- Is this theorem true for **locally** compact spaces ?
- (Folklore) Under $\square_{\aleph_1}$ there is a locally countably compact non metrizable space X of cardinality $\aleph_2$ s.t. all $Y \in [X]^{\leq \aleph_1}$ are metrizable.

Reflection theorems on metrizability

Theorem 1 (A. Dow, 1988)

For a countably compact topological space X,

if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

Theorem 2 (Balogh, 2002)

Assume **Axiom R** *. Then, for a* **locally** *countably compact topological space X,*

if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

Reflection theorems on metrizability

Theorem 3 (S.F., Juhász, Soukup, Szentmiklóssy and Usuba[1], 2010)

Assume **FRP** *. Then, for a* **locally** *countably compact topological space* X*,*

if all subspaces Y *of* X *cardinality* $\leq \aleph_1$ *are metrizable then* X *itself is metrizable.*

Corollary 4

The reflection theorem above is consistent with arbitrarily large size of the continuum (and/or with MA).

Characterization of the reflection

Actually, Theorem 3 is optimal in the following sense:

Theorem 5 (S.F., Sakai, Soukup and Usuba, [2], preprint)

FRP *is* **equivalent** *to the following assertion over* ZFC*:*

For any locally countably compact topological space X*,*

if all subspaces Y *of* X *cardinality* $\leq \aleph_1$ *are metrizable then* X *itself is metrizable.*

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Proof of Theorem 5

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

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(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Without loss of generality we may assume that $a_\alpha \subseteq \kappa \setminus Lim(\kappa)$ for all $\alpha \in E$.

Let $X = E^{\kappa}_{\omega} \cup (\kappa \setminus Lim(\kappa))$ be with the Mrowká topology w.r.t. the ladder system $\langle a_\alpha : \alpha \in S \rangle$.

Then X is locally compact and non-metrizable (by Fodor's Lemma).

But all subspaces Y of X of cardinality $< \kappa$ are metrizable (use f as above for $\beta = \sup Y$ with (*)). □ **(Theorem 5)**

Further reflection theorems (1/2)

FRP is also equivalent to the following topological reflection theorems:

> For a locally separable, countably tight space X, if all subspaces Y of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X itself is meta-Lindelöf.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- The reflection of metrizability of Theorem 3 or Theorem 5 is actually a corollary of the assertion above.

> For a T_1 space with point countable base, if all subspaces Y of X of cardinality $\leq \aleph_1$ are left-separated then X itself is left-separated.

(**[4]** S.F.,Left-separated topological spaces under Fodor-type Reflec -tion Principle, RIMS Kokyuroku No.1619 (2008), 32-42.)

- W. Fleissner (1986) proved that the assertion above follows from Axiom R and it is refuted under the negation of the square principle.

Further reflection theorems (2/2)

FRP is also equivalent to the following topological reflection theorem:

> For a countably tight space of local density $\aleph_1$, if all subspaces Y of X of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then X itself is collectionwise Hausdorff.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- Fleissner (1986) proved also the assertion above under Axiom R.

Reflection of openly generatedness

Theorem 6 (S.F. and Rinot [3], to appear)

The following assertion is equivalent to FRP *over* ZFC*:*

A Boolean algebra B is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ contains a club subset.

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - ω-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

¡Gracias por su atención!

This file and the version for the presentation as well as the present versions of [1] $\sim$ [4] are downloadable from:

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Axiom R

- **Axiom R** is the principle asserting that the following $\mathrm{AR}([\kappa]^{\aleph_0})$ holds for all cardinal $\kappa \geq \aleph_2$:

> $(\mathrm{AR}([\kappa]^{\aleph_0}))$ For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and a ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

- For a set X and cardinal κ, $[\mathbf{X}]^\kappa = \{x \in \mathcal{P}(X) : |x| = \kappa\}$. $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined similarly.

- $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ is said to be **ω_1-club** if it is unbounded w.r.t. $\subseteq$ and closed w.r.t. union of $\subseteq$-chain of length ω_1.

Fodor-type Reflection Principle

- **Fodor-type Reflection Principle (FRP)** is the principle asserting that the following $\mathrm{FRP}(\kappa)$ holds for all regular cardinal $\kappa \geq \aleph_2$:

> **$(\mathrm{FRP}(\kappa))$:** *For any stationary $S \subseteq E^\kappa_\omega$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that*
> - $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;
> - *for any* $f : S \cap I \to \kappa$ *s.t.* $f(\alpha) \in g(\alpha) \cap \alpha$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \kappa$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

- $E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$.

Facts 7

(1) FRP follows from (a weakening of) Axiom R.
(2) FRP is consistent with CH (under certain large cardinal axiom)
(3) FRP is preserved under c.c.c. generic extensions.

countably tight meta Lindelöf spaces

- A space X is **countably tight** if, for any $U \subseteq X$ and $x \in \overline{U}$ there is $U' \in [U]^{\aleph_0}$ s.t. $x \in \overline{U'}$.

- A space X is **meta-Lindelöf** if every open cover of X has a point countable refinement which is also an open cover.

Left-separated spaces

- A space X is *left-separated* if there is a well-ordering $<$ of X s.t. each initial segment of X w.r.t. $<$ is a closed subset of X.

Collectionwise Hausdorff spaces

- A space X is **of local density** κ if for very $p \in X$ there is $Y \in [X]^{\leq\kappa}$ s.t. $p \in int(\overline{Y})$.

- A space X is **collectionwise Hausdorff** if any closed discrete subset D of X can be simultaneously separated by disjoint open sets, i.e., if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

Openly generated Boolean algebras

- A Boolean algebra B is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)

▷ $\mathbf{A} \leq_{\rm rc} \mathbf{B}$ $\Leftrightarrow$ A is a relatively complete subalgebra of B $\Leftrightarrow$ A is a subalgebra of B and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of b)).

Theorem 8 (S.F., Heindorf, Shapiro, 1994)

For a Boolean algebra B, the following are equivalent:

(1) *B is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ "*B is projective*" *for any σ-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) *B has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \to [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\,(a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*