Extendibility and Laver-generic large cardinal axioms

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▶ The Algebra Seminar der TU Wien

2025 年 3 月 13 日，14 日 (11:20～12:20, 10:00～11:00 JST) :

▶ 2025 年春季関東集合論セミナー (with a birthday party for Masahiro Shioya)

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Outline

Outline

Large cardinals characterized by elementary embeddings

- Most of the notions of large cardinals, in particular the notions of large cardinals stronger than measurable cardinals, are characterized as critical points of certain elementary embeddings. For example:

▷ A cardinal κ is said to be supercompact if, for any $\lambda > \kappa$, there are classes j, $M \subseteq \mathsf{V}$ s.t. (1) $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$,[1] (2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically: (3) ${}^{\lambda}M \subseteq M$.

▷ The existence of j with the target model M can be considered as a strong reflection property.

- Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing "for all $\lambda > \kappa$" to "for some $\lambda > \kappa$".

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[1] With "$j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$" we denote the circumstance "M is a transitive class, j is an elementary embedding of the class structure $(\mathsf{V}, \in)$ into the class structure $(M, \in)$, and κ is the critical point of j (i.e. $\kappa = \min\{\mu \in \mathsf{Card} : j(\mu) \neq \mu\}$)"

Large cardinals characterized by elementary embeddings (2/3)

▷ A cardinal κ is said to be supercompact if and only if, for any $\lambda > \kappa$, there are classes j, $M \subseteq V$ s.t. (1) $j : V \xrightarrow{\prec}_{\kappa} M$,(2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically: (3) ${}^{\lambda}M \subseteq M$.

- Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing "for all $\lambda > \kappa$" to "for some $\lambda > \kappa$".

▷ Thus, we obtain the definition of super-almost-huge cardinal by replacing (3) with (3)' ${}^{j(\kappa)>}M \subseteq M$ in the definition of supercompactness.

▷ The definition of superhuge cardinal is obtained by replacing (3) with (3)" ${}^{j(\kappa)}M \subseteq M$ in the definition of supercompactness.

A more comprehensive list:

Large cardinals characterized by elementary embeddings (3/3)

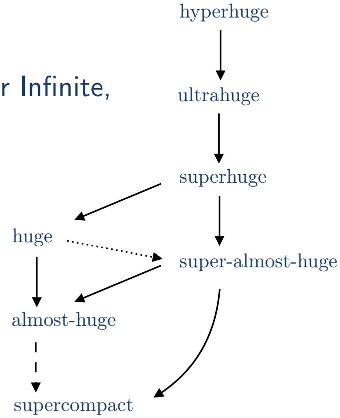
▷ A cardinal κ is said to be supercompact if and only if, for any $\lambda > \kappa$, there are classes j, $M \subseteq V$ s.t. (1) $j : V \xrightarrow{\prec}_{\kappa} M$,(2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically: (3) ${}^{\lambda}M \subseteq M$.

- Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing "for all $\lambda > \kappa$" to "for some $\lambda > \kappa$".

	The condition (3): ${}^{\lambda}M \subseteq M$ replaced by	"for all $\lambda > \kappa$" replaced by "for some $\lambda > \kappa$"
hyperhuge	${}^{j(\lambda)}M \subseteq M$	-
ultrahuge	${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \in M$	-
superhuge	${}^{j(\kappa)}M \subseteq M$	-
super-almost-huge	${}^{j(\kappa)>}M \subseteq M$	-
huge	${}^{j(\kappa)}M \subseteq M$	$\surd$
almost-huge	${}^{j(\kappa)>}M \subseteq M$	$\surd$

Ordered by implication, and by "normal measure one occurrence below"

- [kanamori] The cover of Akihiro Kanamori, The Higher Infinite, Springer Verlag (2004)



B ← − A : "if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$ μ is B in V_κ"

B ←······ A: "if a cardinal κ is with the large cardinal property A, then there are normal measure one many λ with $V_\kappa \models$ "λ is B".

Proposition 1. Suppose that κ is huge. Then,

$$\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is super almost-huge”}\}$$

is a normal measure 1 subset of κ.

Idea of Proof: Modify Theorem 24.11 in

[kanamori] Akihiro Kanamori, The Higher Infinite, Springer Verlag (2004)

to characterize super-almost hugeness. Then solve the corresponding modification of Exercise 24.12 (see these slides for more details).

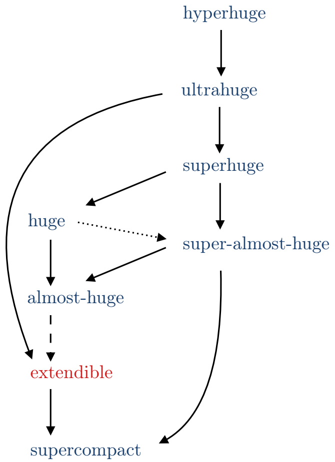
□ (Proposition 1.)

Extendible cardinals

- A cardinal κ is supercompact if , for all $\lambda > \kappa$, there are classes j, $M \subseteq \mathrm{V}$ s.t. (1) $j : \mathrm{V} \xrightarrow{\prec}_{\kappa} M$, (2) $j(\kappa) > \lambda$, and (3) ${}^{\lambda}M \subseteq M$.

	The condition (3): ${}^{\lambda}M \subseteq M$ is replaced by	"for all $\lambda > \kappa$" is replaced by "for some $\lambda > \kappa$"
hyperhuge	${}^{j(\lambda)}M \subseteq M$	-
ultrahuge	${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \in M$	-
superhuge	${}^{j(\kappa)}M \subseteq M$	-
super-almost-huge	${}^{j(\kappa)>}M \subseteq M$	-
huge	${}^{j(\kappa)}M \subseteq M$	$\surd$
almost-huge	${}^{j(\kappa)>}M \subseteq M$	$\surd$
extendible	$V_{j(\lambda)} \in M$	-
supercompact	${}^{\lambda}M \in M$	-

Extendible cardinals (2/3)



B ← – A : "if a cardinal κ is A then there are cofinally many $\mu < \kappa$ s.t. $V_\kappa \models$ "μ is B""

B ←⋯⋯ A: "if a cardinal κ is with the large cardinal property A, then there are normal measure one many $\mu < \kappa$ with $V_\kappa \models$ "μ is B"".

Extendible cardinals (3/3)

Theorem 1a. (A folklore) The following are equivalent:

(a) κ is extendible. I.e., for any $\alpha > \kappa$ there are $\beta \in \mathsf{On}$, and $j : V_\alpha \overset{\prec}{\to}_\kappa V_\beta$ s.t. (*) $j(\kappa) > \alpha$.

(b) κ is Jech extendible. I.e., just as in (a) but (*) is dropped.

(a') For all $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, (*) $j(\kappa) > \lambda$, and $V_{j(\lambda)} \in M$.

(b') Just as in (a') but (*) is dropped.

Super-$C^{(n)}$-extendible cardinals

Theorem 1b. For a cardinal κ and $n \geq 1$, the following are equivalent:

(a) For any $\lambda_0 > \kappa$ there are $\lambda > \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, j_0, and μ s.t. $j_0 : V_\lambda \overset{\prec}{\to}_\kappa V_\mu$, (*) $j(\kappa) > \lambda$, and $V_\mu \prec_{\Sigma_n} \mathsf{V}$.

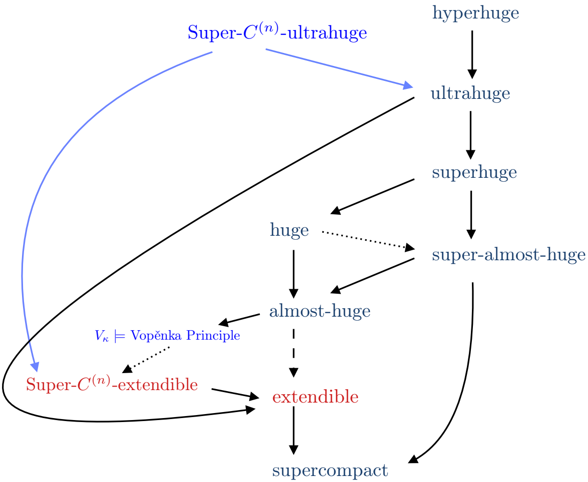
(b) Just as in (a) but without (*).

(a') κ is super-$C^{(n)}$-extendible. I.e., for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, (*) $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$, and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.

(b') Just as in (a') but without (*). □

- In a model M of ZFC, $\kappa \in \mathrm{Card}^M$ is super-$C^{(\infty)}$-extendible if $M \models \bigwedge\!\!\!\bigwedge_{n\in\omega} \kappa$ is super-$C^{(n)}$-extendible.
- Kostas Tsaprounis [tsaprounis] considered super-$C^{(n)}$-extendibility (under a different name) and showed the equivalence of super-$C^{(n)}$-extendibility with $C^{(n)}$-extendibility of Bagaria.

Super-$C^{(n)}$-extendible cardinals (2/2)



B ⟵ A: "if a cardinal κ is with the large cardinal property A, then there are normal measure one many $\lambda < \kappa$ with $V_\kappa \models$ "λ is B" ".

From large cardinals to generic large cardinals

- Small cardinals like $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$ cannot be large cardinals! But they can have many features of large cardinals by being generic large cardinals.

- An important ingredient for the composition of the notion of generic large cardinal is Proposition 22.4 (b) in [kanamori].

  [c.f.: definition of supercompactness]

- For a class $\mathcal{P}$ of p.o.s, κ is said to be $\mathcal{P}$-generic supercompact if, for all $\lambda > \kappa$ there is $\mathbb{P} \in \mathcal{P}$ s.t. for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. (1) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, (2) $j(\kappa) > \lambda$, and $(3)^*$ $j''\lambda \in M$.

- The equivalence in Proposition 22.4 (b) in [kanamori] is no more valid in the generic elementary embedding context but $(3)^*$ is still a closedness property of the target model M. This fact is summarized in Lemma 3.5 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II].

- A small cardinal can be $\mathcal{P}$-generic large cardinal. For example, in the standard model of Proper Forcing Axiom (PFA), $2^{\aleph_0}$ is $\mathcal{P}$-generic supercompact (for $\mathcal{P} =$ proper p.o.s).

From generic large cardinals to Laver-generic large cardinals

- A small cardinal can be $\mathcal{P}$-generic large cardinal. For example, in the standard model of Proper Forcing Axiom (PFA), $2^{\aleph_0}$ is $\mathcal{P}$-generic supercompact (for $\mathcal{P}$ = proper p.o.s).
- Similarly, in the standard model of Martin's Maximum (MM), $2^{\aleph_0}$ is $\mathcal{P}$-generic supercompact (for $\mathcal{P}$ = semi-proper p.o.s).
- Analyzing the standard models of PFA and MM, we obtain the notion of Laver-generic large cardinal:
- A cardinal κ is tightly $\mathcal{P}$-Laver-generic supercompact if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, (4) $j''\lambda \in M$, and (5) $|RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa)$.

  The word "tightly" refers to the condition (5).
- The $\mathcal{P}$-Laver-generic large cardinal axiom for the notion of supercompactness ($\mathcal{P}$-LgLCA for supercompact, for short) is the assertion that $\kappa_{\mathfrak{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$ is tightly $\mathcal{P}$-Laver-generic supercompact cardinal.

Laver-generic large cardinal axioms

- In many cases, the condition "$\kappa = \kappa_{\mathfrak{refl}}$" is a consequence of Laver-gen. supercompactness of κ.

Proposition 2. (Theorem 5.9 in [II]) For $\mathcal{P}$ = σ-closed p.o.s, proper p.o.s, semi-proper p.o.s, ccc p.o.s, etc., if κ is tightly $\mathcal{P}$-Laver gen. supercompact then $\kappa = \kappa_{\mathfrak{refl}}$.

- Along with the hierarchy of large cardinals, we can introduce corresponding LgLCAs (the axioms asserting that $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-gen. large cardinal for the respective notion of large cardinal) by modifying the condition (4) in the definition of tightly $\mathcal{P}$-LgLCA for supercompact.

$\mathcal{P}$-LgLCA for	The condition (4): $j''\lambda \in M$ in case of "super-compact" is replaced by
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$

Laver-generic large cardinal axioms (2/3)

- By definition:

$\mathcal{P}$-LgLCA for hyperhuge

$\downarrow$

$\mathcal{P}$-LgLCA for ultrahuge

$\downarrow$

$\mathcal{P}$-LgLCA for superhuge

$\downarrow$

$\mathcal{P}$-LgLCA for super-almost-huge

$\downarrow$

$\mathcal{P}$-LgLCA for supercompact

B $\longleftarrow$ A : "the axiom A implies the axiom B"

- By Theorem 5.3 in S.F., and T. Usuba [S.F. & Usuba], it follows that $\mathcal{P}$-LgLCA for hyperhuge, and for transfinitely iterable $\mathcal{P}$ is equiconsistent with the existence of a hyperhuge cardinal.

Laver-generic large cardinal axioms (3/3)

∃ a hyperhuge cardinal ⟵ - - - ⟶ $\mathcal{P}$-LgLCA for hyperhuge

Theorem 5.3 in [S.F. & Usuba]

↓

$\mathcal{P}$-LgLCA for ultrahuge

↓

$\mathcal{P}$-LgLCA for superhuge

↓

$\mathcal{P}$-LgLCA for super-almost-huge

↓

$\mathcal{P}$-LgLCA for supercompact

B ⟵ A : "the axiom A implies the axiom B"

B ⟵ - - ⟶ A : "the axioms A and B are equi-consistent."

- $\mathcal{P}$-LgLCA for hyperhuge, for transfinitely iterable $\mathcal{P}$ is one of only few families of strong axioms of set-theory whose exact consistency strength is known.

Super-$C^{(\infty)}$ Laver-generic large cardinal axioms

- Laver-generic version of super-$C^{(\infty)}$ large cardinals can be also introduced. [c.f.: definition of super-$C^{(\infty)}$-extendibility]
  - In contrast to super-$C^{(\infty)}$ large cardinals, super-$C^{(\infty)}$ Laver-generic large cardinal axioms (super-$C^{(\infty)}$-LgLCAs) are first order definable as axiom schemas.
- The super-$C^{(\infty)}$- $\mathcal{P}$-Laver-generic Large Cardinal Axiom for hyperhuge defined as:

The super-$C^{(\infty)}$- $\mathcal{P}$-LgLCA for hyperhuge: For $\kappa = \kappa_{\mathfrak{refl}}$, and for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j''j(\lambda) \in M$, and $|\, RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}}) \,| \leq j(\kappa)$.

LgLCAs imply strong mathematical reflection theorems

Proposition 3. (ccc-LgLCA for supercompact) For any non-free algebra A (in a universal algebraic class of structures) there is non-free subalgebra B of A of size $< 2^{\aleph_0}$.

Proof. Note that ccc-LgLCA for supercompact implies that the continuum is extremely large and hence $\kappa_{\mathfrak{refl}} = 2^{\aleph_0}$.

- Suppose toward a contradiction, that A is a non-free algebra s.t. all subalgebras of A of size $< 2^{\aleph_0}$ are free.
- Let $\lambda := 2^{|A|}$. W.l.o.g., the underlying set of A is $\mu < \lambda$. Let $\mathbb{P}$ be a ccc p.o. adding $\lambda' \geq \lambda$ many reals and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of a ccc p.o. s.t. for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ as in the definition of ccc-LgLCA with $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$ for $\kappa = 2^{\aleph_0}$.
- Then $A \in M$. Since $M \models A \leq j(A)$ and $M \models |A| < j(\kappa) = 2^{\aleph_0}$. By elementarity, it follows that $M \models A$ is free.
- On the other hand, since $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is ccc, $\mathsf{V}[\mathbb{H}] \models A$ is not free. Hence $M \models A$ is not free (see e.g. [fuchino 1992], Theorem 2.1). This is a contradiction,. □ (Proposition 3)

LgLCAs imply strong mathematical reflection theorems (2/3)

Proposition 4. (Cohen-LgLCA for supercompact) Any non-metrizable topological space X with character $< 2^{\aleph_0}$ has a non-metrizable subspace Y of size $< 2^{\aleph_0}$.

Proof. Similarly to Proposition 3. Using a result of Dow, Tall, and Weiss: Cohen forcing preserves non-metrizability of a topological space.

□ (Proposition 3.)

Proposition 5. (1) For any σ-closed generically supercompact cardinal κ, if T is non-special tree then there is $T' \in [T]^{<\kappa}$ which is also non-special.

(2) If σ-closed-LgLCA for supercompact holds, then Rado Conjecture (RC) holds.

(3) If $\mathcal{P}$ contains all ccc p.o.s, then $\mathcal{P}$-LgLCA for supercompact implies ¬RC.

Proof. (1),(2): Similarly to Proposition 3. Using the fact that σ-closed p.o.s preserve non-specialty of trees (Todorčević).

(3): Since MA implies ¬RC and by Theorem 6 below. □ (Proposition 5)

LgLCAs imply strong mathematical reflection theorems (3/3)

Theorem 6. (Theorem 5.7 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II]) ($\mathcal{P}$-LgLCA for supercompact for a stationary preserving $\mathcal{P}$) $\mathsf{MA}^{+<\kappa_{\mathfrak{refl}}}(\mathcal{P})$ holds. □

Corollary 7. Suppose that $\mathcal{P}$ is stationary preserving and contains all σ-closed p.o.s. Then $\mathcal{P}$-LgLCA for supercompact implies the Fodor-type Reflection Principle (FRP).

Proof. By Theorem 6, it follows that $\mathcal{P}$-LgLCA implies $\mathsf{MA}^{+}(\sigma\text{-closed})$. It is known that $\mathsf{MA}^{+}(\sigma\text{-closed})$ implies FRP (See Section 2 of S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness). □ (Corollary 7)

- In contrast:

Proposition 8. FRP is independent over $\mathcal{P}$-LgLCA for supercompact (actually for any large cardinal property) for any class $\mathcal{P}$ of ccc p.o.s as far as the axiom "$\mathcal{P}$-LgLCA for supercompact" is consistent. □

Theorem 6. (Theorem 5.7 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II]) ($\mathcal{P}$-LgLCA for supercompact for stationary preserving $\mathcal{P}$)
$\mathsf{MA}^{+<\kappa_{\mathfrak{refl}}}(\mathcal{P})$ holds.

Theorem 9. ([S.F.& Usuba], see Theorem 6.1 in [S.F.&Gappo&Parente]) ($\mathcal{P}$-LgLCA for ultrahuge) The restricted version of Recurrence Axiom $\underbrace{(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma\text{-}\mathsf{RcA}^+}_{\text{a fragment of Maximality Principle for } \mathcal{P} \text{ and } \mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds for $\Gamma =$ conjunctions of Σ_2 and Π_2 formulas.

Theorem 10. (Theorem 7.2 in [S.F.1]) ($\mathcal{P}$-LgLCA for ultrahuge)
The Unbounded Resurrection Axiom for $\mathcal{P}$ of Tsaprounis [Tsaprounis] holds.

Theorem 11. (Theorem 4.10 in [S.F.& Usuba]) (Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for ultrahuge) The Maximality Principle (Hamkins [Hamkins1]) $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$ holds.

Theorem 12. (Theorems 5.2 and 5.3 in [S.F.& Usuba])
($\mathcal{P}$-LgLCA for hyperhuge (for any $\mathcal{P}$)) The bedrock exists and $\kappa_{\mathfrak{refl}}$ is hyperhuge in the bedrock. Note that this implies $\neg$GA. □

Theorem 13. (1) (Proposition 2.8 in [II]) Suppose that κ is $\mathcal{P}$-generically supercompact and all elements of $\mathcal{P}$ are μ-cc for a cardinal μ. Then Singular Cardinal Hypothesis (SCH) above $\max\{2^{<\kappa}, \mu\}$ holds.

(2) (Corollary 5.2 in [S.F.& Usuba]) ($\mathcal{P}$-LgLCA for hyperhuge (for an arbitrary $\mathcal{P}$)) There are class many huge cardinals, and SCH holds above the continuum.

Proof. (1): A modification of the proof of Solovay's theorem on SCH above a supercompact cardinal will do.

(2): By Theorem 12. □ (Theorem 13)

Laver-Generic Maximum

- Under super-$\mathcal{C}^{(\infty)}$- $\mathcal{P}$-LgLCA for hyperhuge, for $\mathcal{P} =$ semi-proper forcing, almost all set theoretic axioms and principles are integrated into the theory either as a theorem from the theory or as property satisfied by a $\mathcal{P}$-ground.

  - Under super-$\mathcal{C}^{(\infty)}$- $\mathcal{P}$-LgLCA for hyperhuge, for $\mathcal{P} =$ semi-proper, Cichoń's Maximum is a theorem which holds in "many" $\mathcal{P}$-grounds.

LgLCAs for extendible

- A cardinal κ is tightly $\mathcal{P}$-Laver generically extendible if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, (1): ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and (2): $|\,RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

- The $\mathcal{P}$-Laver-generic large cardinal axiom for the notion of extendibility ($\mathcal{P}$-LgLCA for extendible, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-generic extendible cardinal.

- A cardinal κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically extendible if, for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $V_{\lambda} \prec_{\Sigma_n} \mathsf{V}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. (3): ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, (1): ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and (2): $|\,RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

- The super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic large cardinal axiom for the notion of extendibility (super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-generic extendible cardinal.

LgLCAs for extendible (2/2)

- Note that, in general, "κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically extendible" is not formalizable in the language of ZF. In contrast, the axiom "super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible" is formalizable in the language of ZF in infinitely many formulas. This is because the axiom refers to the definable cardinal $\kappa_{\mathfrak{refl}}$.

$\mathcal{P}$-LgLCA for	The condition (4): $j''\lambda \in M$ in case of "super-compact" is replaced by
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$

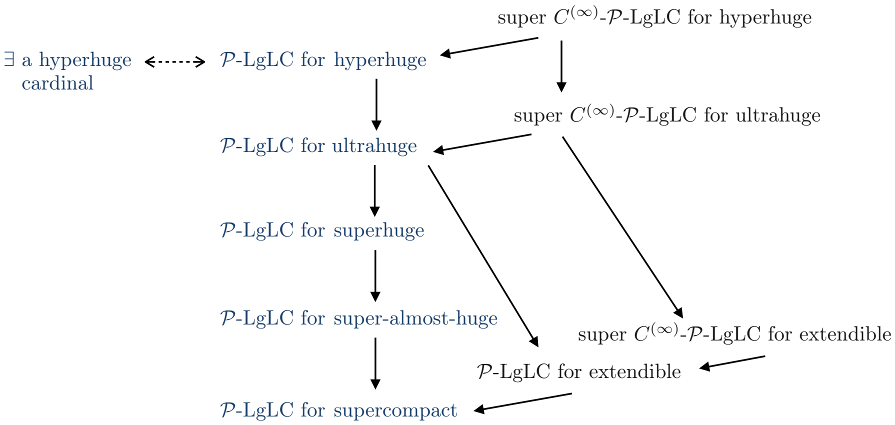
LgLCAs for extendible imply (almost) everything (2/3)

- In Theorems 9 and 11, $\mathcal{P}$-LgLCA for ultrahuge, and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for ultrahuge can be replaced by $\mathcal{P}$-LgLCA for extendible, and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible, respectively.
- In the proof of Theorem 10, it seems that $\mathcal{P}$-LgLCA for ultrahuge is used in its full strength. However we have

Theorem 14. (Improved version of Theorem 7.1 in [S.F.1])
($\mathcal{P}$-LgLCA for extendible)
The Boldface Resurrection Axiom for $\mathcal{P}$ of Hamkins and Johnstone [Hamkins-Johnstone] holds.

- $\mathcal{P}$-LgLCA for extendible has consistency strength below that of an extendible cardinal (see Theorem 15 below).
- Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible have consistency strength strictly less than that of an almost-huge cardinal (see Theorem 16).

LgLCAs for extendible imply (almost) everything (3/3)



B ⟵ A : "the axiom A implies the axiom B"

B ⟵- - -⟶ A : "the axioms A and B are equi-consistent."

Consistency proof of LgLCAs for extendible

Theorem 15. Suppose that κ is extendible. Then, for a transfinitely iterable class $\mathcal{P}$ of p.o.s consisting of stationary preserving p.o.s (e.g. the class of all ccc p.o.s, all σ-closed p.o.s, all proper p.o.s, all semi-proper p.o.s, etc.), there is a p.o. $\mathbb{P}_\kappa \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly $\mathcal{P}$-Laver generic extendible".[2]

Lemma 15.0. If κ is extendible then there are class many measurable cardinals.

Proof. If κ is extendible then it is supercompact (Proposition 23.6 in [kanamori]). Hence, in particular κ is measurable. If $j_0 : V_\gamma \xrightarrow{\prec}_\kappa V_\delta$ with $j_0(\kappa) > \gamma$ then $V_\delta \models$ " there is a normal ultrafilter over $j_0(\kappa)$" by elementarity. Since the normal ultrafilter over $j_0(\kappa)$ in V_δ is really a normal ultrafilter, $j_0(\kappa)$ is measurable. □ (Lemma 15.0)

[Vienna: skip the proof]

[2] The corresponding theorem for the super $C^{(\infty)}$-$\mathcal{P}$-Laver generic extendibility also holds all transfinitely iterable classes $\mathcal{P}$.

Consistency proof of LgLCAs for extendible (2/6)

- We call a mapping $f : M \to N$ cofinal (in N) if for all $b \in N$ there is $a \in M$ s.t. $b \in f(a)$.

Lemma 15.1. (A special case of Lemma 6 in [S.F. & Sakai]) For any cardinal θ and $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N$, letting $N_0 = \bigcup j_0{}''\mathcal{H}(\theta)$, we have $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N_0$ and j_0 is cofinal in N_0. □

Lemma 15.2. (A special case of Lemma 7 in [S.F. & Sakai]) For any regular cardinal θ and cofinal $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N$, there are j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to} M$, $N \subseteq M$, and $j_0 \subseteq j$. □

Lemma 15.3. For a cardinal κ, the following are equivalent:
(a) κ is extendible. (b) For all $\lambda > \kappa$, there are j, M s.t. $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$, $j(\kappa) > \lambda$ and $V_{j(\lambda)} \in M$.

Proof. (b) $\Rightarrow$ (a) is trivial. The other direction follows from Lemma 15.0, Lemma 15.1, and Lemma 15.2. □ (Lemma 15.3)

Consistency proof of LgLCAs for extendible (3/6)

Lemma 15.4. An extendible cardinal κ admits a Laver-function. I.e., there is a mapping $f : \kappa \to V_\kappa$ s.t. for any x, and $\lambda > \kappa$ there are j, M s.t. $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$ s.t. $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $j(f)(\kappa) = x$.[3]

Proof. A modification of the proof of Theorem 20.21 in [Millennium book] (Th. Jech, Set Theory, The Third Millennium Edition) will do.

- Assume, toward a contradiction, that there is no Laver function $f : \kappa \to V_\kappa$.
- ▷ Let $\varphi(f)$ be the formula

$$\exists \underline{\alpha} \exists \underline{\delta} \exists \underline{x} \,(f : \underline{\alpha} \to V_{\underline{\alpha}} \ \wedge\ \underline{\alpha} < \underline{\delta} \ \wedge\ \underline{\delta} \text{ is inaccessible } \wedge\ \underline{x} \in V_{\underline{\delta}} \\ \wedge\ \forall \underline{\delta}' \forall \underline{j} \,((\underline{j} : V_{\underline{\delta}} \overset{\preceq}{\to} V_{\underline{\delta}'} \ \wedge\ \underline{j} \text{ is cofinal in } V_{\underline{\delta}'}) \ \to\ \underline{j}(f)(\underline{\alpha}) \neq \underline{x}))$$

- ▷ If $\varphi(f)$ holds then the witness of $\underline{\alpha}$ in $\varphi(f)$ is uniquely determined. In this case, let δ_f and x_f be witnesses for $\underline{\delta}$ and $\underline{x}$ in $\varphi(x)$. Let $\mu_f := rank(x_f)$. We choose δ_f, x_f and μ_f so that δ_f minimal among the possible witnesses of $\underline{\delta}$ and x_f is chosen so that μ_f is minimal. ▷ If $\varphi(f)$ does not hold, we let $\delta_f := 0$ and $\mu_f := 0$.

Consistency proof of LgLCAs for extendible (4/6)

- $\oplus$ By assumption, we have $\varphi(f)$ for all $f : \kappa \to V_\kappa$.
- ▶ Let ν be an inaccessible cardinal $\geq \max\{\delta_f, \mu_f : f : \alpha \to V_\alpha$ for inaccessible $\alpha \leq \kappa\}$.
- ▷ Let $j^* : V \overset{\prec}{\to}_\kappa M$ be s.t. $(1^\dagger)$: $j^*(\kappa) > \nu$ and $(2^\dagger)$: $V_{j(\nu)} \in M$.
- ▷ Let $A := \{\alpha < \kappa : \forall f\,(f : \alpha \to V_\alpha \ \to\ \varphi(f))\}$.
- ▶ By assumption, $V \models$ "$\forall f\,((f : \kappa \to V_\kappa) \ \to\ \varphi(f))$". By $(2^\dagger)$, it follows
- ▷ $M \models$ "$\forall f\,((f : \kappa \to V_\kappa) \ \to\ \varphi(f))$". Thus we have $M \models j^*(A) \ni \kappa$.
- ▶ Let $f^* : \kappa \to V_\kappa$ be defined by $f^*(\alpha) := \begin{cases} x_{f^* \restriction \alpha}, & \text{if } \alpha \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$
- ▶ Let $x^* := j^*(f^*)(\kappa)$. ▷ By definition of f^*, by $\oplus$, and since $j^*(f^*) \restriction \kappa = f^*$, x^* together with δ_{f^*} and μ_{f^*} witnesses $\varphi(f^*)$. (x^* may be different from x_{f^*} but this does not matter.)
- ▶ In particular, $x^* \neq (j^* \restriction V_{\delta_{f^*}})(f^*)(\kappa) = j^*(f^*)(\kappa) = x^*$. This is a contradiction. □ (Lemma 15.4)

[3] Lemma 15.4 is well-known. See e.g. [corraza]. The present proof is designed so that it can be easily modified to a proof of Laver function for super-$C(\infty)$-extendible which we need in the proof of Theorem 16 .

Consistency proof of LgLCAs for extendible (5/6)

A (sketch of a) proof of Theorem 15: ▶ We show the Theorem for the case that $\mathcal{P}$ is the class of all proper p.o.s. ▶ Let f be a Laver function for extendible cardinal κ (f exists by Lemma 15.2).

▶ Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an CS-iteration of elements of $\mathcal{P}$ s.t.

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \in \mathcal{P} \text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing}, & \text{otherwise.} \end{cases}$$

▶ We show that $\Vdash_{\mathbb{P}_\kappa}$ “$\mathcal{P}$-LgLCA for extendible”.

▷ First, note that $\Vdash_{\mathbb{P}_\kappa}$ “$\kappa = 2^{\aleph_0} = \kappa_{\mathfrak{refl}}$” by definition of $\mathbb{P}_\kappa$.

▷ Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. In $\mathsf{V}[\mathbb{G}_\kappa]$, suppose that $\mathbb{P} \in \mathcal{P}$ and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name for $\mathbb{P}$.

▷ Suppose that $\lambda > \kappa$. By Lemma 15.0, there is an inaccessible $\lambda^* > \lambda$. Let $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$ be s.t. (1^*): $j(\kappa) > \lambda^*$, (2^*): $V_{j(\lambda^*)} \in M$ and (3^*): $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$. (This is possible since f is a Laver function for extendible κ.)

Consistency proof of LgLCAs for extendible (6/6)

- In M, there is a $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$ and $\underset{\sim}{\mathbb{Q}}$ is the direct limit of CS-iteration of small p.o.s in $\mathcal{P}$ of length $j(\kappa)$, and $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{Q}} \sim j(\mathbb{P}_\kappa)$", ▷ By (2^*), the same situation holds in V.
- ▷ We have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \;\sim\; \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. *Here, we are identifying $\underset{\sim}{\mathbb{Q}}$ with corresponding $\mathbb{P}$-name.*
- Let $\mathbb{H}$ be $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter with $\mathbb{G}_\kappa \subseteq \mathbb{H}$.
- The lifting $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \overset{\prec}{\rightarrow}_\kappa M[\mathbb{H}]$; $\underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j(\underset{\sim}{a})[\mathbb{H}]$ witnesses that $\kappa = (\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathbb{G}_\kappa]}$ is tightly $\mathcal{P}$-Laver generic extendible. For this, it suffices to show:

Claim 15.5 $V_\alpha{}^{\mathsf{V}[\mathbb{H}]} \in M[\mathbb{H}]$ for all $\alpha \leq j(\lambda)$.

⊢ By induction on $\alpha \leq j(\lambda)$. The successor step from $\alpha < j(\lambda)$ to $\alpha + 1$ can be proved by showing that $\mathbb{P}_\kappa$-names of subsets of $V_\alpha{}^{\mathsf{V}[\mathbb{H}]}$ can be chosen as elements of M. This is possible because of (2^*). ⊣ (Claim 15.5.)

□ (Theorem 15)

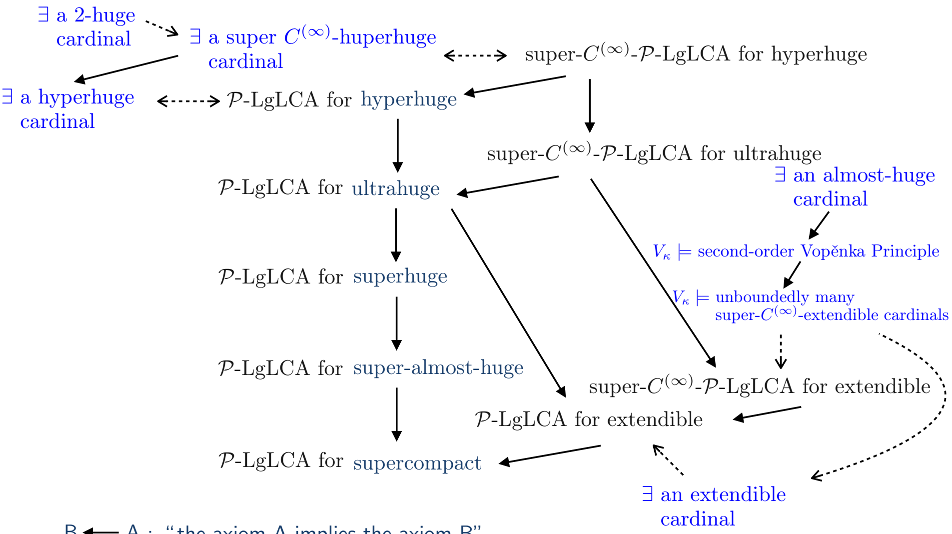
Consistency strength of super-$C^{(\infty)}$-LgLCAs

- The consistency strength of a super-$C^{(\infty)}$ extendible cardinal is below that of an almost huge cardinal (actually, second-order Vopěnka Principle is enough).

- Arguments similar to those for Theorem 15 also works for super-$C^{(\infty)}$-extendible cardinals and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible.

Theorem 16. For a transfinitely iterable class $\mathcal{P}$ consisting of stationary preserving p.o.s, a model of super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible can be obtained starting from a model with a super-$C^{(\infty)}$ extendible cardinal. □

Consistency strength of super-$C^{(\infty)}$-LgLCAs (2/2)



B ⟵ A : “the axiom A implies the axiom B”
B ⟵⋯⟶ A : “the axioms A and B are equi-consistent.”
B ⟵⋯ A : “the consistency of A implies the consistency of B but not the other way around.”

Qual der Wahl

- If we would have decided that Forcing Axioms are correct extensions of ZFC, then the natural choice should be MM or one of the "plus" versions of MM, since, as their names suggest, they are maximal in each types of the variations of forcing axioms.

- Similar argument is not possible for LgLCAs. Two LgLCAs corresponding to two different classes of p.o.s can be incompatible to each other even though the class of p.o.s for one of the axioms is a subclass of the class of p.o.s for the other axiom.

Theorem 17. (1) ([II]) $\mathcal{P}_i$-LgLCA for supercompact for $\mathcal{P}_0 =$ σ-closed p.o.s, $\mathcal{P}_1 =$ proper p.o.s (or semi-proper p.o.s), $\mathcal{P}_2 =$ ccc p.o.s, are mutually incompatible.

(2) (Gappo and Lietz) proper-LgLCA for extendible and semi-proper-LgLCA for extendible are incompatible.

Proof. (1): $\mathcal{P}_0$-LgLCA (for s.c.) $\vdash$ CH; $\mathcal{P}_1$-LgLCA $\vdash 2^{\aleph_0} = \aleph_2$; $\mathcal{P}_2$-LgLCA $\vdash 2^{\aleph_0}$ is very large. (2): proper-LgLCA for extendible $\vdash \delta^1_2 < \omega_2$; semi-proper-LgLCA for extendible $\vdash \delta^1_2 = \omega_2$; ◫ (Theorem 17)

Some further remarks

- LgLCAs for extendible were suggested to me by Gabe Goldberg in 2024 when he visited Kobe after the RIMS conference.
- Super-$C^{(n)}$-extendible cardinal is equivalent with $C^{(n)}$-extendible cardinal (K. Tsaprounis). We examine a simplified proof of this fact by Andreas Lietz in the second Tsukuba talk.
- Andreas Lietz recently obtained also some results which suggest that LgLCAs for extendible and their variants are very much prominent nodes in the tree of possible extensions of ZFC.
- **Open question.** Does the [fuchino-usuba] theorem on the existence of the bedrock follow form the existence of tightly $\mathcal{P}$-generic-extendible cardinal for a reasonable $\mathcal{P}$?

Thank you for your attention!
ご清聴ありがとうございました.
Vielen Dank für die Aufmerksamkeit.

At Shioya's birthday celebration on 13 March 2025. From left to right: Katuya Eda, Masahiro Shioya, Hiroshi Sakai, and me.