

No small forcing adds a new extendible cardinal

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- ▷ Outline
- ▷ Background
- ▷ Proof of Proposition 3
- ▷ Background of background
- ▷ The super- $\mathcal{C}^{(\infty)}$ - \mathcal{P} -LgLCA, and super- $\mathcal{C}^{(\infty)}$ -LgLCAA for hyperhuge
- ▷ Hamkins' Theorem on “adding no large cardinal in extensions”
- ▷ Extendible Cardinals
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(0) LC is definable by a single formula,

A notion LC of large cardinals (measurable, supercompact, $C^{(n)}$ -supercompact, super $C^{(n)}$ -supercompact, ... etc) is regular if (1) all LC-cardinals are inaccessible (2) if λ is not LC and $|P| < \lambda$ then $\prod_{P} \lambda$ is not LC.

For $m \in \mathbb{N}$

$$C^{(m)} := \{ \kappa \in O_n : \forall \lambda \prec_{\Sigma_m} \kappa \}$$

$C^{(n)}$ -supercompact cardinals are different from cardinals which are $C^{(n)}$ and supercompact:

κ is $C^{(n)}$ -supercompact if κ is supercompact with supercompact elementary embeddings j whose target $j(\kappa)$ is $C^{(n)}$ (in V).

Ex 1 For each $n \in \mathbb{N}$, " $C^{(n)}$ and inaccessible" is a regular notion of large cardinals.

Proposition 3 Suppose that (P, ϕ) - RcA^* holds for an attenuating iterable class \mathcal{P} of posets. If LC is a regular notion of large cardinals and LC-cardinal exists, then there are class many LC cardinals.

- ▶ Proof of Proposition 3
- ▶ (P, S) - RcA^*
- ▶ For the details of Ex.1 see: Corollary 2.4 in: Sakaé Fuchino, Maximality and resurrection in light of Laver-generic large cardinal axioms, preprint.
- ▶ Background of background

For an iterable \mathcal{P} and a set S ,

(\mathcal{P}, S) - \mathcal{R} - \mathcal{A}^{*+} : For any \mathcal{L}_E -formula $\varphi = \varphi(\vec{x})$
 and $\vec{a} \in S$, if there is a poset
 $\mathbb{R} \in \mathcal{P}$ s.t. $\mathbb{H}_{\mathbb{R}} \models \varphi(\vec{a})$, there is
 \mathcal{P} - \mathcal{A} -ground W of V s.t. $\vec{a} \in W$
 $W \models \varphi(\vec{a})$ and the machine to
return to V has the size s.t.
there is no inaccessible cardinal below
it. in V

If W is a ground of V and $\mathbb{R} \in W$ and \mathbb{H} is
 s.t. \mathbb{H} is (W, \mathbb{R}) -generic and $V = W[\mathbb{H}]$, then
 we call \mathbb{R} (and/or \mathbb{H}) the machine to return to V
 (from W).

Proof of Proposition 3

No new extendible (7/24)

Proof We work in $ZFC + (\mathcal{P}, \beta)\text{-PCA}^*$
for alternating iterable \mathcal{P} .
Assume that the proposition does not hold. Then
there is some regular notion LC of large cardinals
p.t. there are LC cardinals but they are only set many.
Let λ_0 be the supremum of these LC-cardinals.
Let $\mathbb{P} \in \mathcal{P}_{\beta}$ as in the def. of alternatingness for this λ_0 .
Since LC is a regular notion of large cardinals
 $\mathbb{H}_{\mathbb{P}}$ there is no LC-cardinal.

► Proposition 3

► Background of background

Proposition 3 is a substitute of the following theorem in the small large cardinals setting. Note that the assumption $(\mathcal{P}, \emptyset)\text{-RcA}^*$ of Proposition 3 for a transfinately iterable \mathcal{P} can be forced starting from L with Mahlo cardinals.

- Theorem 3A.** (S.F., and T.Usuba ^[1]) (1) Suppose that \mathcal{P} is a iterable class of posest and the super- $C^{(\infty)}$ - \mathcal{P} -LgLCA for hyperhuge holds. Then $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))\text{-RcA}^{*+}$ holds, ^[2] and for all $n \in \mathbb{N}$, there are stationarily many super- $C^{(n)}$ -hyperhuge cardinals.
- (2) Suppose that the super- $C^{(\infty)}$ -LgLCAA for hyperhuge holds. Then $(\text{all}, \mathcal{H}(\aleph_1))\text{-RcA}^{*+}$ holds, and for all $n \in \mathbb{N}$, there are Bf stationarily many super- $C^{(n)}$ -hyperhuge cardinals.

^[1] On Recurrence Axioms, Annals of Pure and Applied Logic, Vol.176, (10), (2025).
Theorem 4.10, Theorem 5.2, and Corollary 4.2.

^[2] $\kappa_{\text{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$.

2. Hamkins' Theorem on "adding no large cardinal in an extension"

MAIN THEOREM 3 in: Joel Hamkins,

Extensions with the approximation and cover properties have no new large cardinals,

Fundamenta Mathematicae 180 (2003)

By this theorem, we see immediately that measurable, supercompact C^{fm} -supercompact, superhuge etc. are regular.

↑ This was known earlier and called Lévy-Solovay theorem

The following is a (possibly new, but most probably folklore) characterization of extendible cardinals.

Theorem TFAE: (a) κ is extendible.

(b) κ is Jech-extendible.

(a') $\forall \alpha$ all $\lambda > \kappa$ there are $j, M \subseteq V$ s.t. $j: V \xrightarrow{\kappa} M$
 $j(\kappa) > \lambda$, $\kappa > \mu \subseteq M$, $V_{j(\alpha)} \in M$.

(b') Like (a') but without " $j(\kappa) > \lambda$ ".

A characterization of Extendible cardinal (2/3)

No new extendible (14/24)

Hamkins' Main Theorem is applicable to (a') of Theorem 4³ to show that extendibility is a regular notion of large cardinals.

Sketch of $(a) \Rightarrow (a')$

For this implication, it is enough to prove the following two lemmas:

Lemma 5 For a cardinal θ and $j_0 : H(\theta) \overset{\sim}{\rightarrow} N$ for a transitive N , let $N_0 := \bigcup j_0'' H(\theta)$. Then

$j_0 : H(\theta) \overset{\sim}{\rightarrow} N_0$ and j_0 is critical in N_0

A Sketch of the proof of Lemma 6

No new extendible (16/24)

The proof uses a variant of an abstract version of extender
see: S.F., and H.Sakai [1]

Let $\left[\begin{array}{l} \text{we have: } j_0 : H(\theta) \rightarrow N \\ (5) \quad \kappa \rightarrow N \subseteq N \end{array} \right.$

$$\mathcal{M}_j := \{ f \in V : f : \text{dom}(f) \rightarrow V, \text{dom}(f) \in H(\theta) \}$$

$$\mathcal{T} := \{ \langle f, a \rangle : f \in \mathcal{M}_j, a \in j_0(\text{dom}(f)) \}$$

Note that, for $\langle f, a \rangle \in \mathcal{T}$ then $a \in N$

[1] Sakaé Fuchino, and Hiroshi Sakai, The first-order definability of generic large cardinals, submitted.

► Details of the proof is to be found in the: [Handout of this talk.](#)

A Sketch of the proof of Lemma 6 (2/8)

No new extendible (17/24)

For $\langle f, a \rangle, \langle g, b \rangle \in \Pi$ let

$$\langle f, a \rangle \sim \langle g, b \rangle \iff \langle a, b \rangle \in j_0(S_{f(x)=g(y)})$$

where $S_{f(x)=g(y)} := \{ \langle x, y \rangle \in \text{dom}(f) \times \text{dom}(g) : f(x) = g(y) \}$

$$\langle f, a \rangle E \langle g, b \rangle \iff \langle a, b \rangle \in j_0(S_{f(x) \in g(y)})$$

where $S_{f(x) \in g(y)} := \{ \langle x, y \rangle \in \text{dom}(f) \times \text{dom}(g) : f(x) \in g(y) \}$

Claim 1 (1) \sim is an equivalence relation of Π

(2) \sim is congruent to E .

A Sketch of the proof of Lemma 6 (3/8)

No new extendible (18/24)

By "Claim 1" we may consider $(\Pi/\sim, E)$

(Scott's trick)

We denote $\langle f, a \rangle/\sim$ for the class of $\langle f, a \rangle$ and

we simply write $\langle f, a \rangle/\sim \in \langle g, b \rangle/\sim$ if $\langle f, a \rangle \in \langle g, b \rangle$

Claim 2 (Łoś's Lemma) For any \mathcal{L} -formula $\varphi = \varphi(x_0, \dots, x_n)$

and $\langle f_0, a_0 \rangle, \dots, \langle f_{n-1}, a_{n-1} \rangle \in \Pi$, we have

$$\langle \Pi/\sim, E \rangle \models \varphi(\langle f_0, a_0 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim)$$

$$\Leftrightarrow (a_0, \dots, a_n) \in j_0(S_\varphi)$$

where

$$S_\varphi = \left\{ \langle u_0, \dots, u_{n-1} \rangle \in \text{dom}(f_0) \times \dots \times \text{dom}(f_{n-1}) \mid \forall f \in \mathcal{F} \varphi(f(u_0), \dots, f(u_{n-1})) \right\}$$

A Sketch of the proof of Lemma 6 (4/8)

No new extendible (19/24)

For $u \in V$ let $f_u: \mathbb{1} \rightarrow V$ be s.t.

$f_u(\phi) = u$. $\therefore V \rightarrow \mathbb{T}/\sim$; $u \mapsto \langle f_u \phi \rangle / \sim$

Claim 3

i is an elementary embedding of $\langle V, E \rangle$ into $\langle \mathbb{T}/\sim, E \rangle$

Claim 4 (1) E well-founded

(2) E is set like

A Sketch of the proof of Lemma 6 (5/8)

No new extendible (20/24)

By Claims 3, 4 there is a Mostowski collapse

$$m : \langle \pi/n, E \rangle \rightarrow \langle V, \in \rangle$$

Let $M := m''\pi/n$ and $j = m \circ i$

$$\langle V, \in \rangle \xrightarrow[\cong]{i} \langle \pi/n, E \rangle \xrightarrow[\cong]{m} \langle M, \in \rangle$$

Claim 5 $j \upharpoonright H(\theta)^V = j_0$ in particular,

$$j : V \xrightarrow[\cong]{\cong} M$$

A Sketch of the proof of Lemma 20 (8/8)

No new extendible (23/24)

Then $\langle b_i : i < \omega \rangle$ is a descending sequence
in $\langle j_0(Q), j_0(\mathbb{C}) \rangle$

Since well-foundedness is Δ_1

$N \models j_0(\langle Q, \mathbb{C} \rangle)$ is non well-founded

$(j_0 : H(\theta) \xrightarrow{\downarrow} N)$ so

$H(\theta) \models \langle Q, \mathbb{C} \rangle$ is non-well-founded.

But a witness of non well-foundedness generates an
infinite descending \leftarrow chain.

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Thank you for your attention!

ご清聴ありがとうございました。

Vielen Dank für die Aufmerksamkeit.