No small forcing adds a new extendible cardinal

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2026 年 5 月 27 日 (15:30～17:00 JST) : **Kobe Set Theory Seminar**

2026 年 6 月 8 日 (16:00～16:30 JST) : **Yamaguchi Set Theory seminar 2026**

The following slides are typeset using upLaTeX with beamer class, and presented on UP2 Version 2.0.2

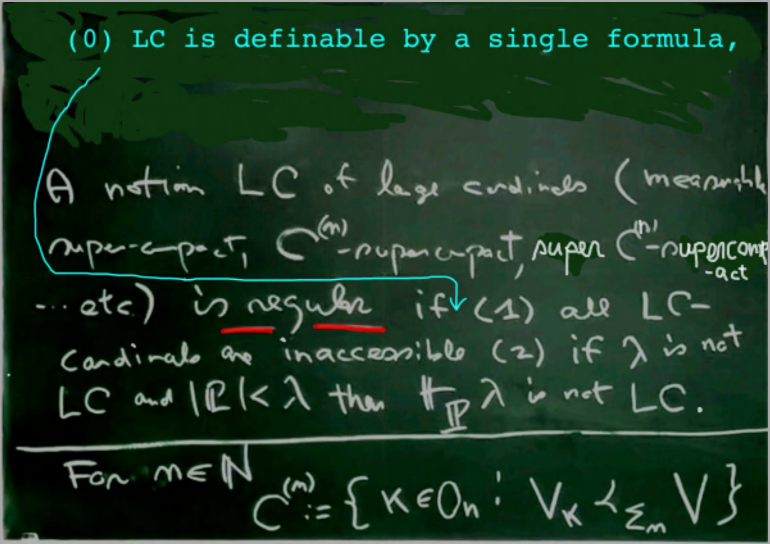
The most up-to-date version of these slides is downloadable as

https://fuchino.ddo.jp/slides/yamaguchi2026-slides-pf.pdf

Handout of the talk: https://fuchino.ddo.jp/talks/yamaguchi-2026-memo-x.pdf

No small forcing adds a new extendible cardinal — Outline

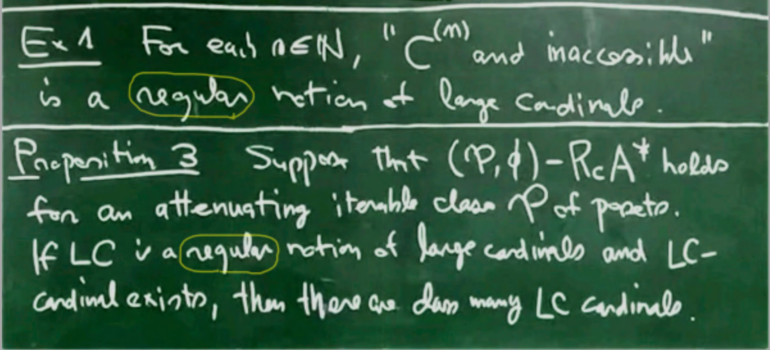
Background


(0) LC is definable by a single formula,
A notion LC of large cardinals (measurable
super-compact, C^(n)-supercompact, super C^(n)-supercompact
... etc) is regular if (1) all LC-
cardinals are inaccessible (2) if λ is not
LC and |P| < λ then ⊩_P λ is not LC.
For n ∈ N
C^(n) := {κ ∈ On : V_κ ≺_{Σ_n} V}

Background (2/4)

$C^{(n)}$-supercompact cardinals are different from cardinals which are $C^{(n)}$ and supercompact:

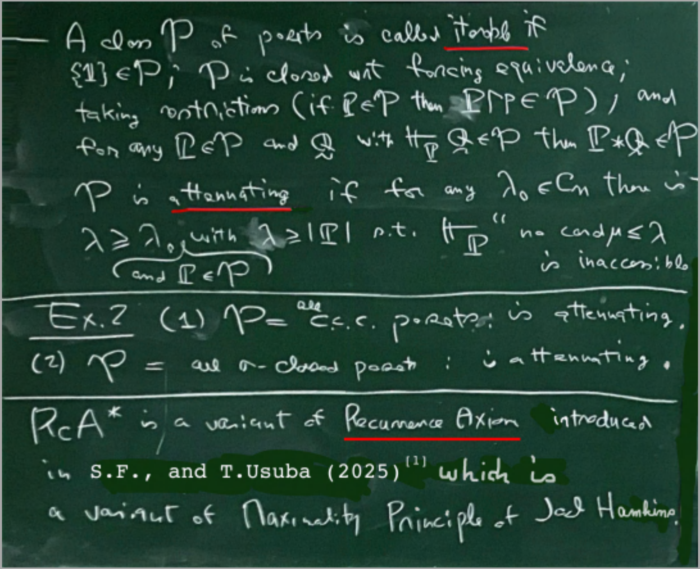
κ is $C^{(n)}$-supercompact if κ is supercompact with supercompact elementary embeddings j whose target $j(\kappa)$ is $C^{(n)}$ (in V).



▶ Proof of Proposition 3 ▶ $(\mathcal{P}, S)$-RcA* ▶ For the details of Ex.1 see: Corollary 2.4 in: Sakaé Fuchino, Maximality and resurrection in light of Laver-generic large cardinal axioms, preprint.

▶ Background of background

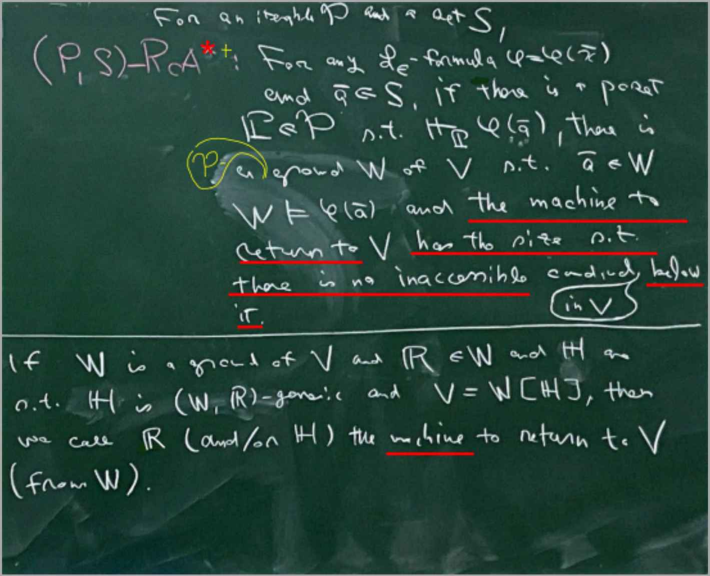
Bcackground (3/4)



[1] Sakaé Fuchino, and Toshimichi Usuba, On recurrence axioms, APAL, Vol.176, (10), (2025).

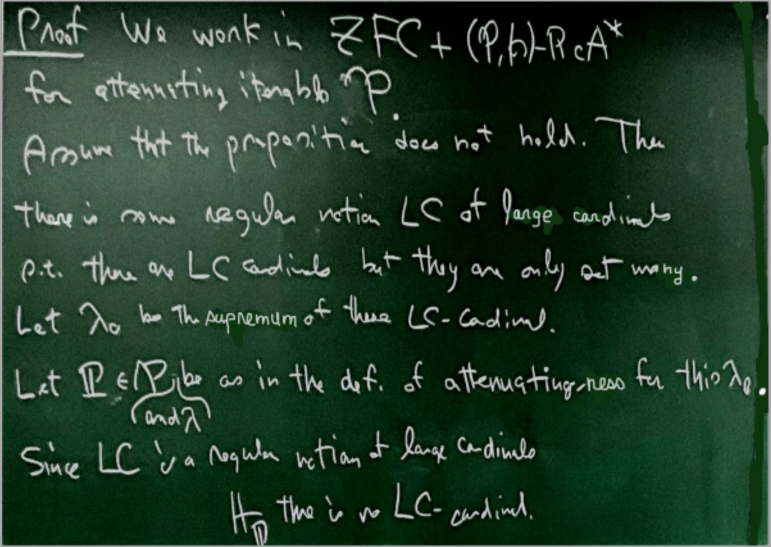
▶ Proposition 3

Background (4/4)



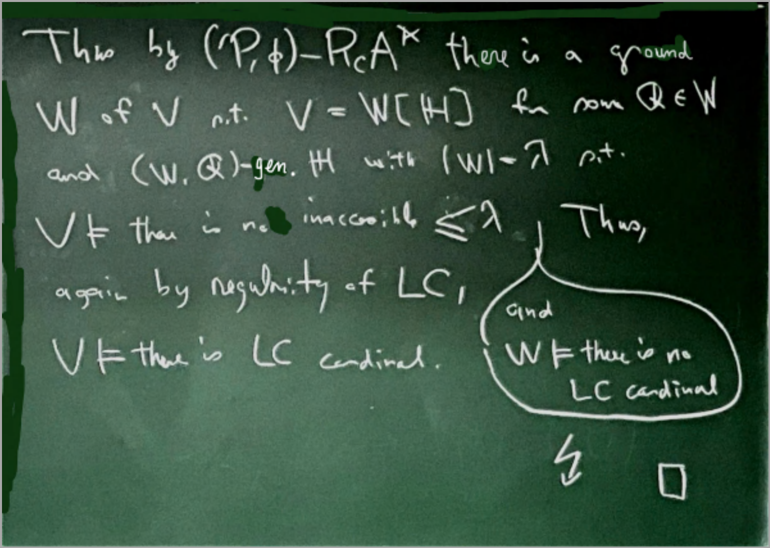
▶ Proposition 3 ▶ Background of background

Proof of Proposition 3



▶ Proposition 3 ▶ Background of background

Proof of Proposition 3 (2/2)



▶ Proposition 3

Proposition 3 is a substitute of the following theorem in the small large cardinals setting. Note that the assumption $(\mathcal{P}, \emptyset)$-RcA* of Proposition 3 for a transfinitely iterable $\mathcal{P}$ can be forced starting from L with Mahlo cardinals.

Theorem 3A. (S.F., and T.Usuba [1]) (1) Suppose that $\mathcal{P}$ is a iterable class of posest and the super- $C^{(\infty)}$- $\mathcal{P}$-LgLCA for hyperhuge holds. Then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA^{*+} holds,[2] and for all $n \in \mathbb{N}$, there are stationarily many super-$C^{(n)}$-hyperhuge cardinals.

(2) Suppose that the super-$C^{(\infty)}$-LgLCAA for hyperhuge holds. Then $(all, \mathcal{H}(\aleph_1))$-RcA^{*+} holds, and for all $n \in \mathbb{N}$, there are Bf stationarily many super-$C^{(n)}$-hyperhuge cardinals.

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[1] On Recurrence Axioms, Annals of Pure and Applied Logic, Vol.176, (10), (2025). Theorem 4.10, Theorem 5.2, and Corollary 4.2.

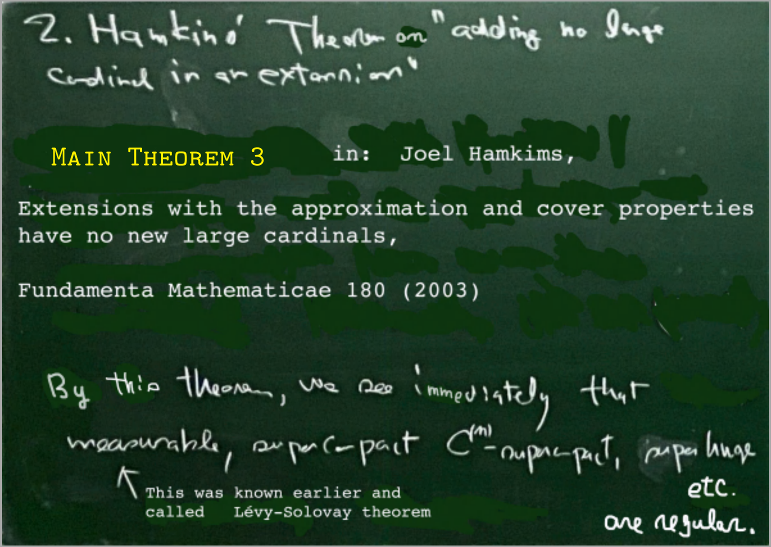
[2] $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$.

The super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge: For all $n \in \mathbb{N}$, for any $\lambda_0 > 2^{\aleph_0}$ and p.o. $\mathbb{P} \in \mathcal{P}$, there are $\lambda > \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with (a) $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa_{\mathfrak{refl}}} M$, (b) $j(\kappa_{\mathfrak{refl}}) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, (c) (tightness) $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa_{\mathfrak{refl}})$, and $(\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\})$
(d) $j''j(\lambda) \in M$ (the closure property corresponding to hyperhugeness),
(e) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$; $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$ (i.e. $\lambda \in (C^{(n)})^{\mathsf{V}}$; $j(\lambda) \in (C^{(n)})^{\mathsf{V}[\mathbb{H}]}$).

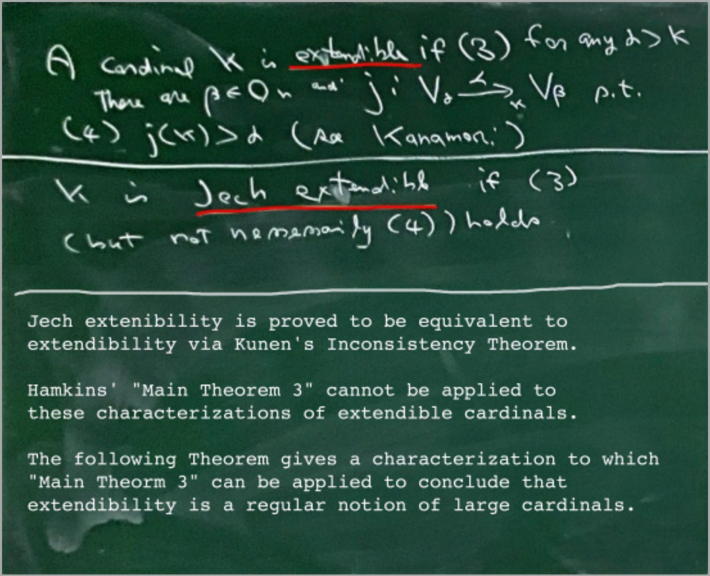
The super-$C^{(\infty)}$-LgLCAA for hyperhuge: For all $n \in \mathbb{N}$, for any $\lambda_0 > 2^{\aleph_0}$ and p.o. $\mathbb{P}$, there are $\lambda > \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a p.o. s.t. for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with (a) $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{2^{\aleph_0}} M$, (b) $j(2^{\aleph_0}) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, (c) (tightness) $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(2^{\aleph_0})$, and
(d) $j''j(\lambda) \in M$ (the closure property corresponding to hyperhugeness),
(e) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$; $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$ (i.e. $\lambda \in (C^{(n)})^{\mathsf{V}}$; $j(\lambda) \in (C^{(n)})^{\mathsf{V}[\mathbb{H}]}$).

▶ Background of background

Hamkins' Theorem on "adding no large cardinal in extensions"



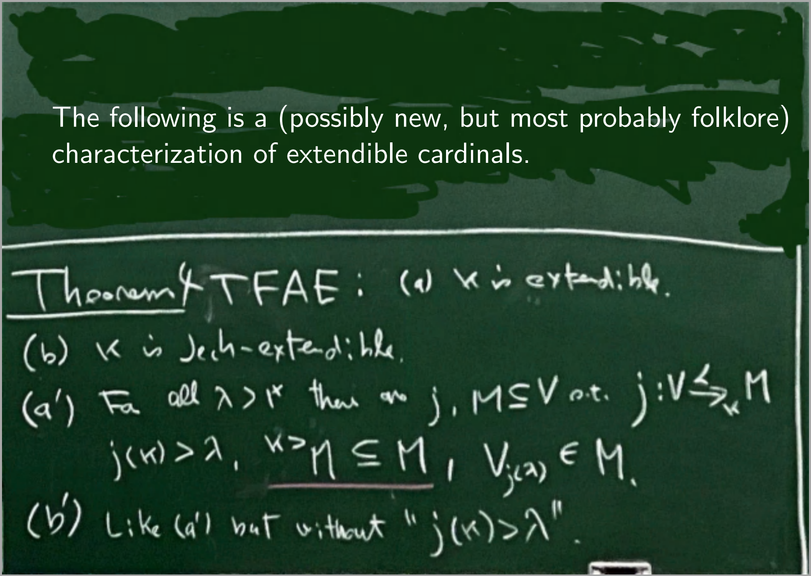
Extendible Cardinals



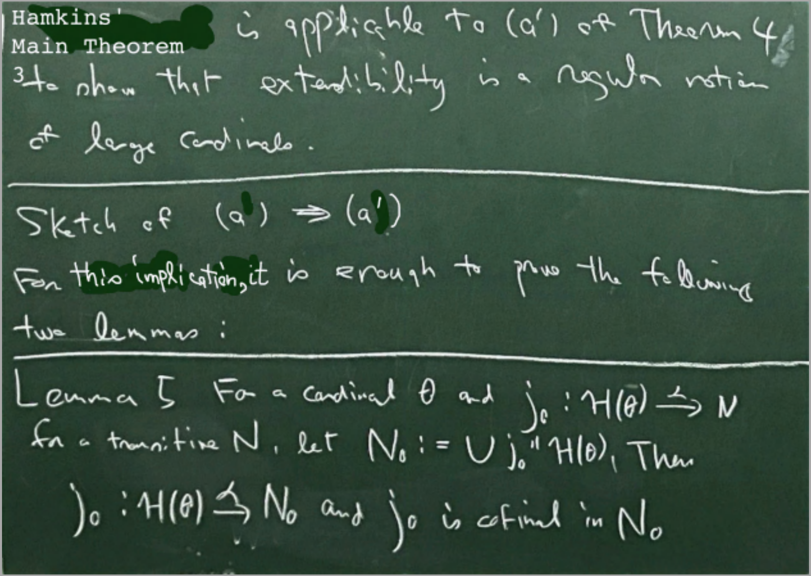
- Hamkins' Main Theorem 3

A characterization of Extendible cardinal

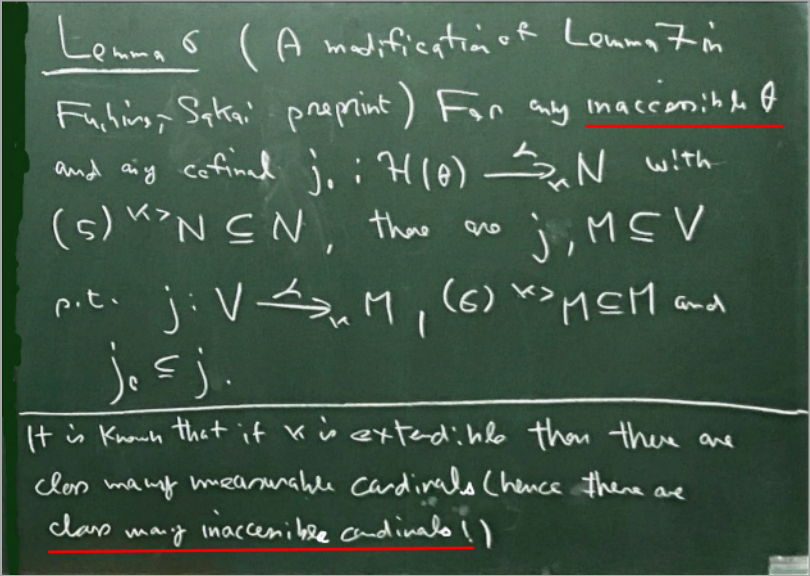
The following is a (possibly new, but most probably folklore) characterization of extendible cardinals.



A characterization of Extendible cardinal (2/3)



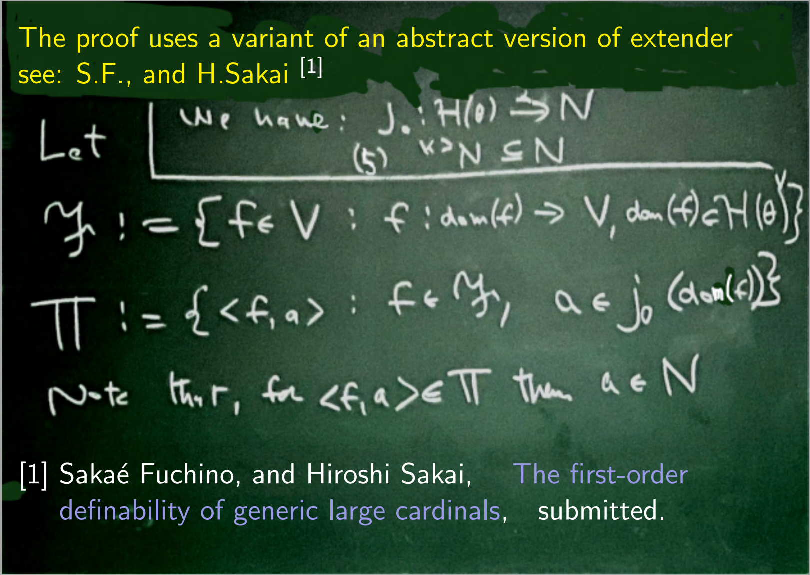
A characterization of Extendible cardinal (3/3)



▶ Proof of (6)

A Sketch of the proof of Lemma 6

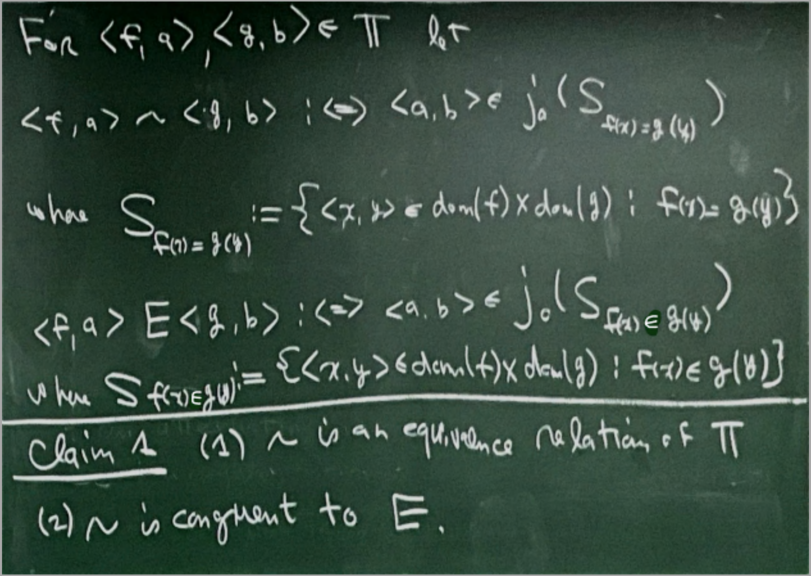
The proof uses a variant of an abstract version of extender
see: S.F., and H.Sakai [1]



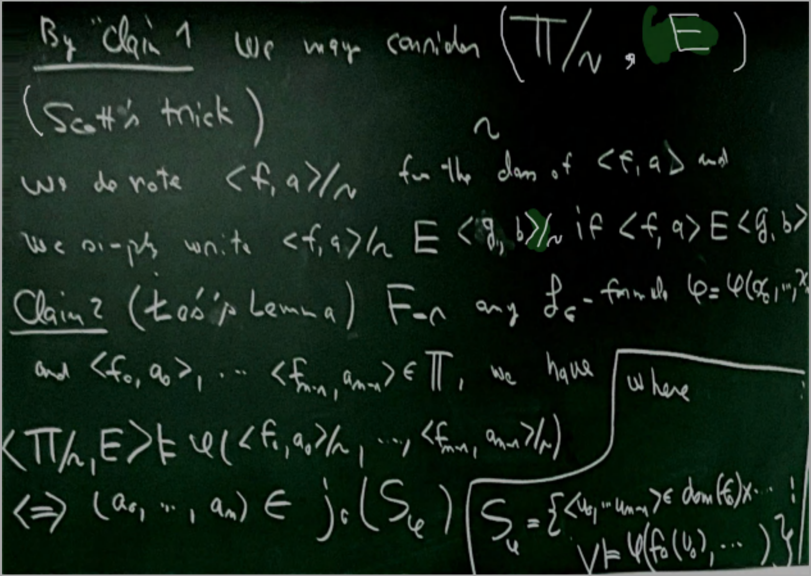
[1] Sakaé Fuchino, and Hiroshi Sakai, The first-order definability of generic large cardinals, submitted.

- Details of the proof is to be found in the: Handout of this talk.

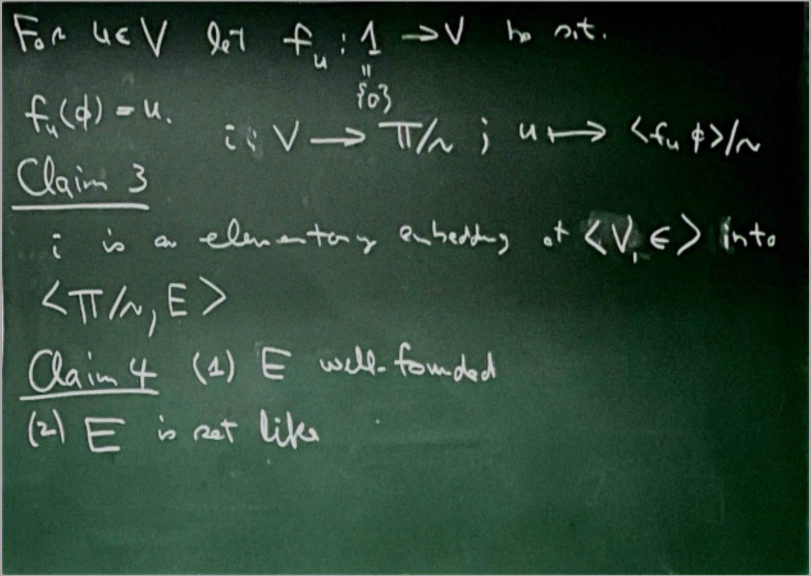
A Sketch of the proof of Lemma 6 (2/8)



A Sketch of the proof of Lemma 6 (3/8)

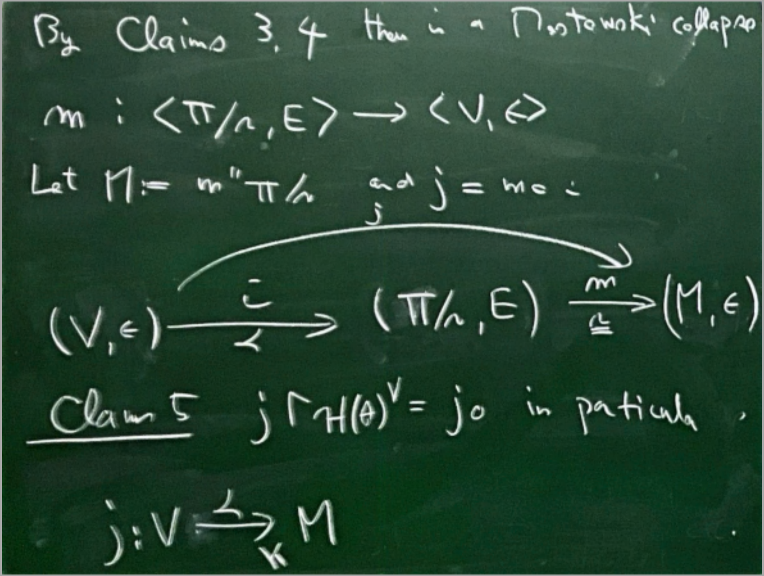


A Sketch of the proof of Lemma 6 (4/8)

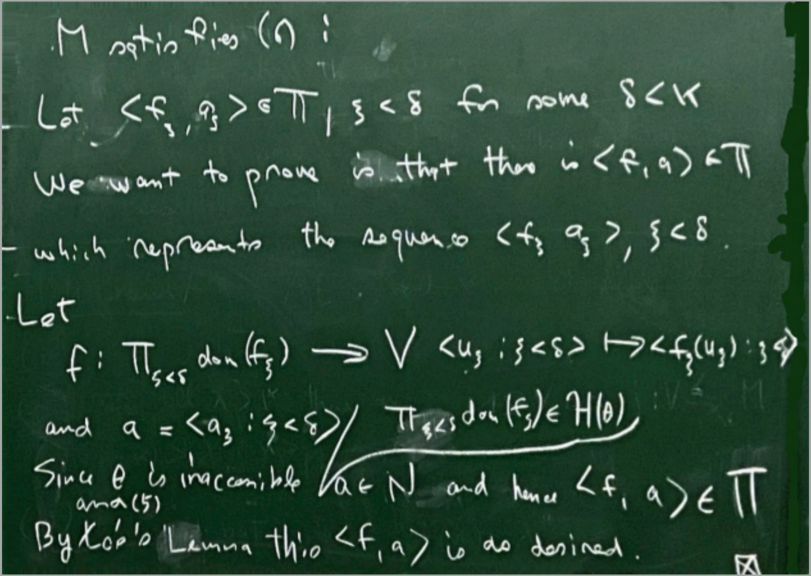


▶ Proof of Claim 4, (1)

A Sketch of the proof of Lemma 6 (5/8)

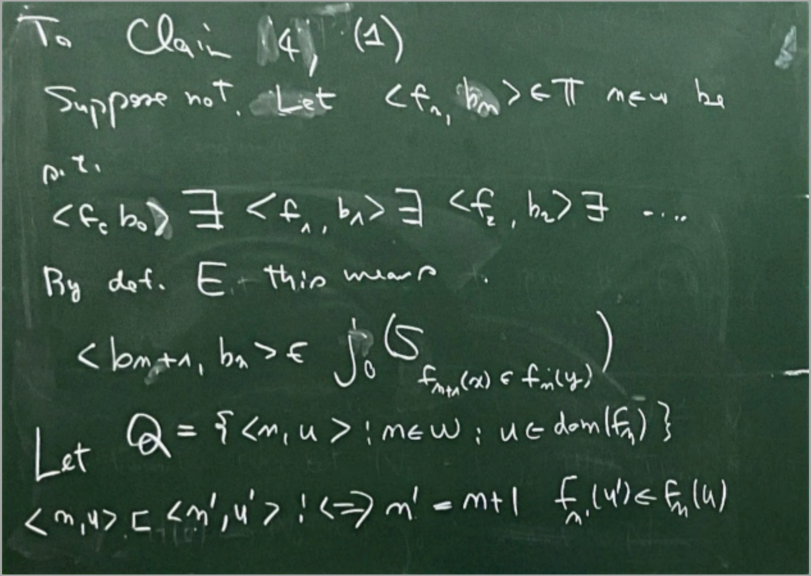


A Sketch of the proof of Lemma 6 (6/8)



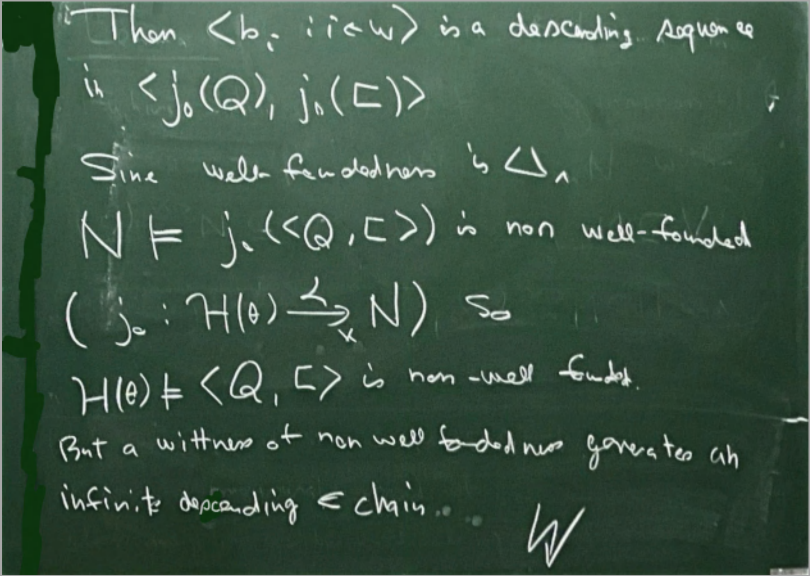
▶ the condition (6)

A Sketch of the proof of Lemma 6 (7/8)



- Claim 4, (1)

A Sketch of the proof of Lemma 20 (8/8)



Thank you for your attention!

ご清聴ありがとうございました.

Vielen Dank für die Aufmerksamkeit.