The trichotomy of the possible size of the continuum under the existence of a Laver-generic large cardinal

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Definition 0.1 *For a cardinal* κ *and a class* $\mathcal{P}$ *of posets, we call* κ *a* Laver-generically supercompact for $\mathcal{P}$ *if, for any* $\lambda \geq \kappa$ *and any* $\mathbb{P} \in \mathcal{P}$*, there are a poset* $\mathbb{Q} \in \mathcal{P}$ *with* $\mathbb{P} \lessdot \mathbb{Q}$ *and* $(\mathsf{V}, \mathbb{Q})$*-generic filter* $\mathbb{H}$ *such that there are an inner model* $M \subseteq \mathsf{V}[\mathbb{H}]$ *and a class function* $j \subseteq \mathsf{V}[\mathbb{H}]$ *with*

(0.1) $\quad j : \mathsf{V} \xrightarrow{\preceq} M$,

(0.2) $\quad crit(j) = \kappa,\ j(\kappa) > \lambda$,

(0.3) $\quad \mathbb{P},\ \mathbb{H} \in M$ *and*

(0.4) $\quad j''\lambda \in M$.

A cardinal κ is called Laver-generically superhuge (super almost-huge, resp.), if the conditions in Definition 0.1 holds with (0.4) replaced by

(0.5) $\quad j''j(\kappa) \in M \quad (j''\mu \in M$ for all $\mu < j(\kappa)$, resp.).

A cardinal κ is tightly Laver-generically supercompact (superhuge, super almost-huge, resp.) if we have

(0.6) $\quad |\mathbb{Q}| = j(\kappa)$

in addition for the poset $\mathbb{Q}$ in the definition of Laver-generial supercompactness (superhugeness, super almost-hugeness, resp.)

Lemma 0.1 (Lemma 2.4 in [2]) *Suppose that* $\mathbb{G}$ *is a* $(\mathsf{V}), \mathbb{P}$*-generic filter for a poset* $\mathbb{P} \in \mathsf{V}$ *and* $j : \mathsf{V} \xrightarrow{\prec} M \subseteq \mathsf{V}[\mathbb{G}]$ *such that, for cardinals* κ, λ *in* V *with* $\kappa \leq \lambda$, $crit(j) = \kappa$ *and* $j''\lambda \in M$*.*

(1) *For any set* $A \in \mathsf{V}$ *with* $\mathsf{V} \models |A| \leq \lambda$*, we have* $j''A \in M$*.*

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$*.*

(3) *For any* $A \in \mathsf{V}$ *with* $A \subseteq \lambda$ *or* $A \subseteq \lambda^2$ *we have* $A \in M$*.*

(4) $(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$*, Thus, if* $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$*, then* $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$*.*

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$*.*

(6) $j \restriction A \in M$ *for all* $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$*.*

Theorem 0.2 (Theorem 6.2 in [2]) (1) *Suppose that* ZFC + *"there exists a supercompact cardinal (super almost-huge cardinal, superhuge cardinal, resp.)" is consistent. Then* ZFC + *"there exists a tightly Laver-generically supercompact cardinal (super almost-huge cardinal, superhuge cardinal, resp.) for* σ*-closed posets" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a tightly Laver-generically super almost-huge cardinal" is consistent as well.*

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal (superhuge cardinal, resp.)" is consistent. Then* ZFC + *"there exists a tightly Laver-generically supercompact cardinal (super almost-huge cardinal, resp.) for ccc posets" is consistent as well.* ❑

Proposition 0.3 (1) (Lemma 6.3 in [2]) *Suppose that* κ *is generically measurable by a* ω_1 *preserving* $\mathbb{P}$*. Then* $\kappa > \omega_1$*.*

(2) (Lemma 6.4 in [2]) *Suppose that* κ *is Laver-generically supercompact for* ω_1*-preserving* $\mathcal{P}$ *with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$*. Then we have* $\kappa = \omega_2$*.*

(3) (Lemma 6.5 in [2]) *Suppose that* $\mathcal{P}$ *is a class of posets containing a poset* $\mathbb{P}$ *such that any* $(\mathsf{V}, \mathbb{P})$*-generic filter* $\mathbb{G}$ *codes a new real. If* κ *is a Laver-generically supercompact for* $\mathcal{P}$*, then* $\kappa \leq 2^{\aleph_0}$*.*

(4) (Lemma 6.6 in [2]) *Suppose that* $\mathcal{P}$ *is a class of posets such that elements of* $\mathcal{P}$ *do not add any reals. If* κ *is generically supercompact by* $\mathcal{P}$*, then we have* $2^{\aleph_0} < \kappa$*.*

(5) (Proposition 2.7 in [2]) *Suppose that* κ *is generically supercompact for a class* $\mathcal{P}$ *of posets such that all* $\mathbb{P} \in \mathcal{P}$ *are* μ*-cc for some* $\mu \in Card$*. Then*

(a) SCH *holds above* $\max\{2^{<\kappa}, \mu\}$*.*

(b) *For all regular* $\lambda \geq \kappa$*, there is a* μ*-saturated normal fine filter over* $\mathcal{P}_\kappa(\lambda)$*.*

(6) (Theorem 6.8 in [2]) *If* κ *is tightly Laver-generically superhuge for ccc posets, then* $\kappa = 2^{\aleph_0}$*.*

For a class $\mathcal{P}$ of posets and cardinals μ, κ, we consider the following strengthening of the forcing axiom for $\mathcal{P}$:

$\mathsf{MA}^{+\mu}(\mathcal{P},<\kappa)$: *For any* $\mathbb{P}\in\mathcal{P}$*, any family* $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}|<\kappa$ *and any family* $\mathcal{S}$ *of* $\mathbb{P}$*-names such that* $|\mathcal{S}|\leq\mu$ *and* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of ω_1" *for all* $\underset{\sim}{S}\in\mathcal{S}$*, there is a* $\mathcal{D}$*-generic filter* $\mathbb{G}$ *over* $\mathbb{P}$ *such that* $\underset{\sim}{S}[\mathbb{G}]$ *is a stationary subset of* ω_1 *for all* $\underset{\sim}{S}\in\mathcal{S}$*.*

For a poset $\mathbb{P}$, $\mathbb{P}$-name $\underset{\sim}{S}$ of a set of subsets of On and a filter $\mathbb{G}$ on $\mathbb{P}$, let

$$\underset{\sim}{S}(\mathbb{G})=\{b\,:\,b=\{\alpha\in\mathsf{On}\,:\,\mathbb{p}\Vdash_{\mathbb{P}}\text{“}\check{\alpha}\in\underset{\sim}{s}\text{”}\text{ for a }\mathbb{p}\in\mathbb{G}\}\text{ for a }\mathbb{P}\text{-name }\underset{\sim}{s}\text{ such that }\Vdash_{\mathbb{P}}\text{“}\underset{\sim}{s}\in\underset{\sim}{S}\text{ and }\sup(\underset{\sim}{s})\equiv\sup(b)\text{”}\}. \tag{0.7}$$

Note that if $\mathbb{G}$ is a $(\mathsf{V},\mathbb{P})$-generic filter, then $\underset{\sim}{S}(\mathbb{G})=\underset{\sim}{S}[\mathbb{G}]$.

For uncountable cardinals μ and $\kappa>\aleph_1$, let $\mathsf{MA}^{++\mu}(\mathcal{P},<\kappa)$ be the strengthening of $\mathsf{MA}^{+\mu}(\mathcal{P},<\kappa)$ defined by:

$\mathsf{MA}^{++\mu}(\mathcal{P},<\kappa)$: *For any* $\mathbb{P}\in\mathcal{P}$*, any family* $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}|<\kappa$ *and any family* $\mathcal{S}$ *of* $\mathbb{P}$*-names such that* $|\mathcal{S}|\leq\mu$ *and* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$" *for some* $\omega<\eta_{\underset{\sim}{S}}\leq\theta_{\underset{\sim}{S}}\leq\mu$ *with* $\eta_{\underset{\sim}{S}}$ *regular, for all* $\underset{\sim}{S}\in\mathcal{S}$*, there is a* $\mathcal{D}$*-generic filter* $\mathbb{G}$ *over* $\mathbb{P}$ *such that* $\underset{\sim}{S}(\mathbb{G})$ *is stationary in* $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ *for all* $\underset{\sim}{S}\in\mathcal{S}$*.*

Clearly $\mathsf{MA}^{++\omega_1}(\mathcal{P},<\kappa)$ is equivalent to $\mathsf{MA}^{+\omega_1}(\mathcal{P},<\kappa)$.

Theorem 0.4 (Theorem 6.7 in [2]) *For an arbitrary class* $\mathcal{P}$ *of posets, if* $\kappa>\aleph_1$ *is a Laver-generically supercompact for* $\mathcal{P}$*, then* $\mathsf{MA}^{++\mu}(\mathcal{P},<\kappa)$ *holds for all* $\mu<\kappa$*.*

For principles "$\mathsf{SDLS}\cdots$" mentioned below, see [1] and [2].

Theorem 0.5 (Theorem 6.9 in [2]) (1) *Suppose that* κ *is Laver-generically supercompact for* σ*-closed posets. Then* $2^{\aleph_0}=\aleph_1$*,* $\kappa=\aleph_2$*,* $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ *and hence also* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat},<\aleph_2)$ *holds.*

(2) *Suppose that* κ *is Laver-generically supercompact for proper posets. Then* $2^{\aleph_0}=\kappa=\aleph_2$*,* $\mathsf{PFA}^{+\omega_1}$ *and hence also* $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat},<2^{\aleph_0})$ *holds.*

(3) *Suppose that* κ *is Laver-generically supercompact for ccc posets. Then* $2^{\aleph_0}\geq\kappa$ *and* $\mathcal{P}_\kappa(\lambda)$ *for any regular* $\lambda\geq\kappa$ *carries an* $\aleph_1$*-saturated normal ideal. In particlular,* κ *is* κ*-weakly Mahlo.* $\mathsf{MA}^{++\mu}(ccc,<\kappa)$ *for all* $\mu<\kappa$*,* $\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat},<\kappa)$ *and* $\mathsf{SDLS}^{int}(\mathcal{L}^{PKL}_{stat},<\kappa)$ *also hold.*

References

[1] Sakaé Fuchino, André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, preprint.
http://fuchino.ddo.jp/papers/SDLS-x.pdf

[2] akaé Fuchino, André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II, in preparation.
`http://fuchino.ddo.jp/papers/SDLS-II-x.pdf`