Suslin Hypothesis under Laver-generic Large Cardinal Axioms[(0)]

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渕野 昌 (Sakaé Fuchino)

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▶ A (definable) class $\mathcal{P}$ of posets is said to be *iterable* if (a): $\{\mathbb{1}\} \in \mathcal{P}$, (b): $\mathcal{P}$ is closed with respect to forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$), (c): closed with respect to restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and (d): for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

▶ For a (two step) iterable class $\mathcal{P}$ of posets, and a notion LC of large cardinal, κ is said to be *tightly $\mathcal{P}$-Laver generic* LC (*$\mathcal{P}$-Lg. LC*, for short) if:

(e) for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and, for $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\preceq}{\to}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\tilde{\mathbb{P}} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, M satisfies C'_{LC} which is the generic large cardinal variant of the closedness property C_{LC} associated with the notion LC of large cardinal, and $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \le j(\kappa)$.

LC	C'_{LC}
* hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$
* extendible	$V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]} \in M$
* supercompact	$j''\lambda \in M$
* measurable	$j''\kappa \in M$ but without the details about "λ"

▶ An iterable class $\mathcal{P}$ of posets is *almost attenuating* if $\mathcal{P}$ provably satisfies at least one of the following two properties: (f): $\mathcal{P}$ contains $\mathbb{P} \in \mathcal{P}$ that collapses ${\omega_2}^{\mathsf{V}}$; (g): $\mathcal{P}$ contains a poset $\mathbb{P}$ that adds a new real.

▶ $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$.

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[0] https://fuchino.ddo.jp/talks/kobe-26-07-15-cheat-sheet.pdf

Lemma 1 (Lemma 2.4 in S.F. [1]) (1) *If $\kappa_{\mathfrak{refl}}$ is a tightly $\mathcal{P}$-Lg. extendible for an iterable class $\mathcal{P}$ of posets, then $\mathcal{P}$ is stationary preserving. I.e., elements of $\mathcal{P}$ preserve the stationarity of stationary subsets of ω_1 in* V.

(2) *Suppose that $\mathcal{P}$ is an almost attenuating iterable class of posets, $\mathcal{P}$ is $\aleph_1$-preserving, and a cardinal κ is tightly $\mathcal{P}$-Lg. supercompact. If (g) holds then $\kappa = 2^{\aleph_0}$.*
If (f) holds, then $\kappa = \aleph_2$. In this case, if no elements of $\mathcal{P}$ adds a new real, then we have $2^{\aleph_0} = \aleph_1$; if (g) also holds in addition, then we have $\aleph_2 = 2^{\aleph_0}$. Note that, in both cases, we have $\kappa = \kappa_{\mathfrak{refl}}$.

(2′) *Suppose that $\mathcal{P}$ is the class of all posets and κ is a tightly $\mathcal{P}$-Lg. supercompact cardinal. Then $\kappa = \aleph_1 = 2^{\aleph_0}$.*

(3) *Suppose that $\mathcal{P}$ is an iterable class of posets which are cardinal and cofinality preserving, and provably satisfies (g). If κ is tightly $\mathcal{P}$-*LgLCA *for supercompact then $\kappa = 2^{\aleph_0}$, and it is weakly hyper-Mahlo (and more).* ❑

► Lemma 1 above motivates the definition of following axioms:

$\mathcal{P}$-Laver generic Large Cardinal Axiom for LC ($\mathcal{P}$-LgLCA *for* LC, for short):
$\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Lg. for LC.

► By Lemma 1, (1), an iterable class $\mathcal{P}$ of posets must be stationary preserving, if $\mathcal{P}$- LgLCA for extendible should be consistent.

Laver generic Large Cardinal Axiom for all posets and for LC ($\mathcal{P}$- LgLCAA *for* LC, for short):
$2^{\aleph_0}$ is tightly $\mathcal{P}$-Lg. for LC for $\mathcal{P}$ = all posets.

Remark 2 *For a transfinitely iterable almost attenuating and stationary preserving $\mathcal{P}$, and a sufficiently large $n \in \mathbb{N}$, we obtain a model of $\mathbb{P}$-*LgLCA *for* LC *by starting from a model with $C^{(n)}$-*LC *κ by iterating along with a Laver-function for $C^{(n)}$-*LC *(the choice of n can be optimized according to $\mathcal{P}$. For example we dan set $n = 1$ if $\mathcal{P}$ is Σ_2, see Theorem 5.2 in [2]).*
The same construction also works for all posets $\mathcal{P}$ with finite support iteration of length κ for an LC *κ along with a Laver-funcion for* LC *to establish* LgLCAA *for* LC.

For an iterable class $\mathcal{P}$ of posets, a set A (of parameters), and a set Γ of $\mathcal{L}_\in$- formulas, *$\mathcal{P}$-Recurrence Axiom for formulas in Γ with parameters from A* (with or without +) is the following assertion expressed as an axiom scheme in $\mathcal{L}_\in$:

$(\mathcal{P}, A)_\Gamma$-RcA^+ For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ " $\varphi(\overline{a}^{\checkmark})$ ", then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

Theorem 3 (Theorem 6.1 in S.F., Gappo, and Parente [4] with a slight improvement[1]) *Suppose that κ is tightly $\mathcal{P}$-Lg. extendible for an iterable class $\mathcal{P}$ of posets. Then $(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$-*$\mathsf{RcA}^+$ *holds where Γ is the set of all formulas which are conjunctions of a Σ_2-formula and a Π_2-formula.* ❑

Corollary 4 *Let Γ be the set of formulas as in Theorem 3.* (1) *For an iterable $\mathcal{P}$, $\mathcal{P}$-*LgLCA *for extendible implies $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-*$\mathsf{RcA}^+$.

(2) LgLCAA *for extendible implies $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))$-*$\mathsf{RcA}^+$. ❑

► Theorem 1, (1) follows form Theorem 3 and the following Theorem 5, (1). A part of the rest of Theorem 1 also follows from Theorem 3 and Theorem 5, though there is also a simple direct proof.

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[1] This theorem is stated in [4] for tightly $\mathcal{P}$-Lg. ultrahuge. However the same proof is actually applicable to tightly $\mathcal{P}$-Lg. extendible.

Theorem 5 (Theorem 3.3 in S.F. and Usuba [6]) *Assume that $\mathcal{P}$ is an iterable class of posets.* (1) *If $\mathcal{P}$ contains a poset which adds a real (over the universe), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $\neg$CH.*

(2) *Suppose that $\mathcal{P}$ is almost attenuating and* (f) *holds. Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} \leq \aleph_2$.*

(2') *If $\mathcal{P}$ is almost attenuating and* (f) *holds, then $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$-RcA does not hold.*

(3) *If $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA holds then all $\mathbb{P} \in \mathcal{P}$ preserve $\aleph_1$ and they are also stationary preserving.*

(4) *If $\mathcal{P}$ if almost attenuating, and both* (f) *and* (g) *hold, then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.*

(5) *If $\mathcal{P}$ contains a poset which collapses ${\aleph_1}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies* CH.

(5') *If $\mathcal{P}$ contains a poset which collapses ${\aleph_1}^{\mathsf{V}}$ then $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$-RcA does not hold.*

(6) *Suppose that all $\mathbb{P} \in \mathcal{P}$ preserve cardinals and $\mathcal{P}$ contains posets adding at least κ many reals for each $\kappa \in$ Cn. Then $(\mathcal{P}, \emptyset)_{\Sigma_2}$-RcA$^+$ implies that $2^{\aleph_0}$ is very large. More precisely, we have $2^{\aleph_0} > \aleph_\alpha$ for any Σ_2-definable ordinal α whose definition is absolute for $\mathcal{P}$-generic extensions.*

(6') *Suppose that $\mathcal{P}$ is as in* (6)*. Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-RcA$^+$ implies that $2^{\aleph_0}$ is a limit cardinal. Thus if $2^{\aleph_0}$ is regular in addition, then $2^{\aleph_0}$ is weakly inaccessible.* ❑

Theorem 6 (Theorem 5.7 in [5]) *$\mathcal{P}$-LgLCA for supercompact implies MA$(\mathcal{P}, <\kappa_{\mathfrak{refl}})$ (actually it even implies a strengthening of MA$^{++}(\mathcal{P}, <\kappa_{\mathfrak{refl}})$).*

A sketch of Proof. For $\mathbb{P} \in \mathcal{P}$ and a family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of $\mathbb{P}$ with $|\mathcal{P}| < \kappa_{\mathfrak{refl}}$, let $\lambda \geq |\mathbb{P}|$ and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of an element of $\mathcal{P}$ such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa_{\mathfrak{refl}}} M$, $\mathbb{P}$, $\mathbb{P} * \mathbb{H} \in M$, $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa)$, and $j''\lambda \in M$. Without loss of generality, we may assume that the underlying set of $\mathbb{P}$ is λ. Then $j''\mathbb{P} \in M$ and also $j \restriction \mathbb{P} \in M$. We have $j(\mathcal{D}) = \{j(D) : D \in \mathcal{D}\}$. Thus, for the $\mathbb{P}$ part $\mathbb{G}$, (the filter generated from) $j''\mathbb{G}$ witnesses $M \models$ "there is a $j(\mathcal{D})$-generic filter over $j(\mathbb{P})$".

By elementarity, it follows that $\mathsf{V} \models$ "there is a $\mathcal{D}$-generic filter over $\mathbb{P}$". The last assertion follows from Lemma 1, (1). ❑ (Theorem 6)

► It is known that each of sufficient strengthenings of LgLCAs (super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible for verious $\mathcal{P}$, and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCAA for extendible) implies

Reflection Principles down to $<\kappa_{\mathfrak{refl}}$ (in case of $\mathcal{P}$-LgLCA for each $\mathcal{P}$)
— S.F., Ottenbreit Maschio Rodrigues, and Sakai [5]

Generic Absoluteness — S.F., Gappo, and Parente [4]

Forcing Axiom with double plus and more
— S.F., Ottenbreit Maschio Rodrigues, and Sakai [5]

Maximality Principle — S.F., and Usuba [6]

Resurrection Axiom — S.F. [1]

for corresponding class of posets with maximal possible set of parameters.

Under (super-$C^{(\infty)}$-)$\mathcal{P}$-LgLCA for hyperhuge for verious $\mathcal{P}$, and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCAA for hyperhuge we obtain additionally the Bedrock Axiom — S.F., and Usuba [6].

▷ Since each of these principles and theorems are already very strong, the combination of them should approach almost complete deduction power and decide most of the mathematical assertions which are independent over ZFC. Thus we may expect:

Thesis. Strengthenings of LgLCAs for strong notions of LC decide all reasonable mathematical statements (known to be independent sofar).

► We check how this happens with Suslin's Hypothesis.

▶ In the following we assume κ is regular if not specified otherwise. A *κ-tree* is a tree of hight κ such that all levels of T are of cardinality $<\kappa$.

We assume that a tree $T = \langle T, <_T\rangle$ is growing upwards and it is *normal* (i.e. all elements of limit hight are decided by its predecessors), and in case of κ-tree, we assume that $T \subseteq {}^{\kappa>}\kappa$ or $T \subseteq {}^{\kappa>}\kappa_0$ if $\kappa = (\kappa_0)^+$, and $<_T = \subsetneq$.

$\mathbb{P}_T = \langle T, \mathbb{1}_T, \sqsubset_T\rangle$ is the forcing version of a κ-tree T which is the poset with $\sqsubset_T = (<_T)^{-1}$ and $\mathbb{1}_T = \emptyset$ (see the conventions on κ-trees above). Pairwise incompatibility of t, $t' \in T$, denoted by $t \perp_T t'$ is meant in the sense of the incompatibility with respect to $\sqsubset_T$. We say $A \subseteq T$ is an antichain if it is antichain in the sense of $\mathbb{P}_T$.

A κ-tree T is *well-pruned* if $|T \uparrow t| = \kappa$ for all $t \in T$. T is *ever-branching* if for any $t \in T$ there are $t <_T t'$, t'' such that $t' \perp_T t''$. Thus T is ever-branching if and only if $\mathbb{P}_T$ is atomless.

We shall say T satisfies the κ-c.c., if $\mathbb{P}_T$ does. A κ-tree T is a *κ-Suslin tree* if T is well-pruned, ever-branching, and satisfies the κ-c.c. If κ-tree T is κ-Suslin then it has no branch of length κ since if b is a branch of length κ then we can construct an antichain of size κ by collecting off-branchies of b. An ω_1-Suslin tree is simply called a *Suslin tree*. SH is the assertion that there is no Suslin tree.

A κ-tree is an *Aronszajn tree* if it has no branch of length κ. By the remark above A κ-Suslin tree is an Aronszajn tree.

Lemma 7 (1) *If T is Suslin and $\mathbb{P}$ is productively c.c.c. then $\Vdash_{\mathbb{P}}$ " T is Suslin ".*

(2) *If T is a Suslin tree and $\mathbb{P}$ is σ-closed then $\Vdash_{\mathbb{P}}$ " T is Suslin ". σ-closed can be replaced by $<\omega_1$-strategically closed.*

(3) *If T is a Suslin tree then $\mathbb{P}_T$ is σ-Baire.*

(4) *σ-closed in (2) cannot be replaced by σ-Baire.*

Proof. (1): Suppose that T is a Suslin tree, and $\mathbb{P}$ is productively c.c.c. If the assertion does not hold, then we may assume that $\Vdash_{\mathbb{P}}$ " T has an antichain of size $\aleph_1$ ".

Let $\{\underset{\sim}{t}_\alpha : \alpha < \omega_1\}$ be $\mathbb{P}$-names of $\aleph_1$ elements of such an antichain. We may assume that (h): $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{t}_\alpha \perp_T \underset{\sim}{t}_{\alpha'}$ " for any $\alpha < \alpha' < \omega_1$. For each $\alpha < \omega_1$, let $\mathbb{p}_\alpha \in \mathbb{P}$ and $t_\alpha \in T$ be such that $\mathbb{p}_\alpha \Vdash_{\mathbb{P}}$ " $\underset{\sim}{t}_\alpha = t_\alpha$ ".

Since $\mathbb{P}_T \times \mathbb{P}$ is c.c.c. by the choice of $\mathbb{P}$, there are $\alpha < \beta < \omega_1$, such that $\langle t_\alpha, \mathbb{p}_\alpha\rangle$ and $\langle t_\beta, \mathbb{p}_\beta\rangle$ are compatible. That is, there is $\mathbb{p} \leq_t \mathbb{p}_\alpha$, $\mathbb{p}_\beta$ and t_α and t_β are compatible. But then we should have $\mathbb{p} \Vdash_{\mathbb{P}}$ " $\underset{\sim}{t}_\alpha$ and $\underset{\sim}{t}_\beta$ are compatible ". This is a contradiction to (7).

(2): Suppose that T is a Suslin tree, and $\mathbb{P}$ is σ-closed If the assertion does not hold, then we may assume that $\Vdash_{\mathbb{P}}$ " T has an antichain of size $\aleph_1$ ".

Let $\{\underset{\sim}{t}_\alpha : \alpha < \omega_1\}$ be $\mathbb{P}$-names of $\aleph_1$ elements of such an antichain with the property as in the proof of (1). By σ-closedness we can find a decreasing sequence $\langle \mathbb{p}_\alpha : \alpha < \omega_1\rangle$ in $\mathbb{P}$, and $t_\alpha \in T$, $\alpha < \omega_1$ such that $\mathbb{p}_\alpha \Vdash_{\mathbb{P}}$ " $\underset{\sim}{t}_\alpha = t_\alpha$ ". Then $\{t_\alpha : \alpha < \omega_1\}$ is an antichain in T (in V). This is a contradiction.

If $\mathbb{P}$ is only $<\omega_1$-strategically closed, we can still prove the claim of (2) by a slight modificataion of the argument above.

(3): Suppose that T is a Suslin tree. We prove first the following:

Claim 7.1 *If $D \subseteq T$ is open dense then there is $\alpha_D \in \omega_1$ such that $\{t \in T : ht(t) \geq \alpha_D\} \subseteq D$.*

$\vdash$ Let $A = \{t \in T : t \in D \, \forall \beta < ht(t) \, (t \restriction \beta \notin D)\}$. Then A is an antichain. It follows that $|A| \leq \aleph_0$. Let $\alpha_D = \sup\{ht(t) : t \in A\}$. Then $\alpha_D < \omega_1$ and this is as desired. $\dashv$ (Claim 7.1)

Let D_n, $n \in \omega$ be open dense subsets of T then, letting $\alpha := \sup_{n\in\omega} \alpha_{D_n}$, we have $\{t \in T : ht(f) \geq \alpha\} \subseteq \bigcup_{n\in\omega} D_n$.

(4): $\mathbb{P}_T$ forcies that T is not a Suslin tree. ❑ (Lemma 7)

Theorem 8 (S. Shelah), see Jech [7], Theorem 28.12) *Suppose that* $\mathbb{P} := \mathrm{Fn}(\omega, 2)$*. Then* $\Vdash_{\mathbb{P}}$ "$\neg\mathsf{SH}$"*.*

Proof. Without loss of generality we may replace $\mathrm{Fn}(\omega, 2)$ with $\mathbb{P} := {}^{\omega>}\omega$ with reverse inclusion.

Let $\underset{\sim}{r}$ be the standard $\mathbb{P}$-name of the Cohen real $r := \bigcup \mathbb{G}$ for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$.

In V, let $\langle f_\alpha : \alpha < \omega_1 \rangle$ be a sequence of mapping such that

(i) $f_\alpha : \alpha \to \omega$, f_α is 1-1, and $\omega \setminus \mathrm{rng}(f_\alpha)$ is infinite.

(j) $f_\alpha \subseteq^* f_\beta$ for all $\alpha < \beta < \omega_1$ where $\subseteq^*$ denotes the inclusion mod finite.

Cf. Kunen [8], Theore m 5.9: $T := \{s \in {}^{\omega_1>}\omega : s =^* f_\alpha \text{ for some } \alpha < \omega_1\}$ with $<_T := \subsetneq$ is an ω_1-Aronszajn tree. Here, we also mean with $=^*$ mod finite.

Note that T can never be a Suslin tree (see the next Lemma 9).

In $\mathsf{V}[\mathbb{G}]$, let

$$T_r := \{t \in {}^{\omega_1>}\omega : t = r \circ (f_\alpha \restriction \beta) \text{ for some } \beta \le \alpha < \omega_1\} \quad \text{with} \quad <_{T_r} := \subsetneq.$$

It is enough to show the following:

Claim 8.1 *T_r satisfies the c.c.c.*

$\vdash$ Let $\underset{\sim}{A}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{A} \text{ is an uncountable subset of } T_{\underset{\sim}{r}} \text{”}.$$

Let $\mathbb{p} \in \mathbb{P}$ be arbitrary. Let $\underset{\sim}{B}$ be a $\mathbb{P}$-name of an uncountable subset $\underset{\sim}{B}$ of ω_1, and $\underset{\sim}{g}$ a $\mathbb{P}$-name of a mapping from $\underset{\sim}{B}$ to ω_1 such that $\Vdash_{\mathbb{P}}$ “$\alpha \le \underset{\sim}{g}(\alpha)$ for all $\alpha \in \underset{\sim}{A}$”, and

$$\mathbb{p} \Vdash_{\mathbb{P}} \text{“} \{\underset{\sim}{r} \circ (f_{\underset{\sim}{g}(\alpha)} \restriction \alpha) : \alpha \in \underset{\sim}{B}\} \subseteq \underset{\sim}{A} \text{”}.$$

Since $\mathbb{P} \restriction \mathbb{p}$ is countable, there are $\mathbb{q} \le_{\mathbb{P}} \mathbb{p}$, uncountable $C \subseteq \omega_1$, and mapping $g : C \to \omega_1$ (all in V) such that

$$\mathbb{q} \Vdash_{\mathbb{P}} \text{“} \{\underset{\sim}{r} \circ (f_{g(\alpha)} \restriction \alpha) : \alpha \in C\} \subseteq \underset{\sim}{A} \text{”}.$$

Let $n = \mathrm{dom}(\mathbb{q})$. By Fodor's Lemma and by thinning out C if neccessary, we may assume that $D_\alpha := \left(f_{g(\alpha)} \restriction \alpha\right)^{-1} {}''n$, for $\alpha \in C$ are all the same, and

(k) $\left(f_{g(\alpha)} \restriction \alpha\right) \restriction D_\alpha = f_{g(\alpha)} \restriction D_\alpha$ for $\alpha \in C$ are all the same.

Let $D \in [\omega_1]^{\le n}$ be such that $D = D_\alpha$ for all $\alpha \in C$.

Subclaim 8.1.1 *For any $\alpha_0, \alpha_1 \in C$ with $\alpha_0 < \alpha_1$, there is $\mathbb{r} \le_{\mathbb{P}} \mathbb{q}$ such that*

$$\mathbb{r} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{r} \circ (f_{g(\alpha_0)}) \subseteq \underset{\sim}{r} \circ (f_{g(\alpha_1)}) \text{”}.$$

$\vdash$ Let $\alpha_0, \alpha_1 \in C$ be such that $\alpha_0 < \alpha_1$. Then

(l) $f_{g(\alpha_0)} \restriction \alpha_0 \subseteq^* f_{g(\alpha_1)} \restriction \alpha_1$. Let

$$E_0 := \{\xi < \alpha_0 : f_{g(\alpha_0)} \restriction \alpha_0(\xi) \ne f_{g(\alpha_1)} \restriction \alpha_1(\xi)\}.$$

Then E_0 is a finite set and $E_0 \cap D = \emptyset$ by (k). Let

$$h_0 := (f_{g(\alpha_1)} \restriction \alpha_0)^{-1} \circ (f_{g(\alpha_0)} \restriction \alpha_0), \text{ and}$$
$$h_1 := (f_{g(\alpha_0)} \restriction \alpha_0)^{-1} \circ (f_{g(\alpha_1)} \restriction \alpha_0).$$

Note that h_0 and h_1 are partial functions from α_0 to α_0.

Let E be the closure of E_0 with respect to h_0 and h_1. That is,

$$E := \{(h_0)^i(\xi) : \xi \in E_0,\ i \in \omega \text{ and } h^i(\xi) \text{ is defiined}\}$$
$$\cup \{(h_1)^i(\xi) : \xi \in E_0,\ i \in \omega \text{ and } h^i(\xi) \text{ is defiined}\}.$$

Then E is finite by (l), and $E \cap D = \emptyset$ by (k) and since $f_{g(\alpha_0)}$ and $f_{g(\alpha_1)}$ are both injection.

Thus, by letting $m > n$ be such that $(f_{g(\alpha_0)}{}''E \ \cup \ f_{g(\alpha_1)}{}''E) \subseteq m$,

$$\mathbb{r} := \mathbb{q} \cup \{\langle k, 0\rangle : n \le k < m\}$$

is as desired. $\dashv$ (Subclaim 8.1.1)

$\dashv$ (Claim 8.1)

$\square$ (Theorem 8)

Lemma 9 (see Kunen [8], Exercise (39) in Chapter II) (1) *If T is a subtree of $T_{inj} := \{t \in {}^{\omega_1>}\omega :$ t is 1-1$\}$ (with $<_T = \subsetneq$), T cannot be a Suslin tree.*

(2) *T mentioned in the proof of the previous Theorem 8 can never be a Suslin tree.*

Proof. For $n \in \omega$, Let

$$A_n := \{s \in T : \mathrm{dom}(s) = \alpha + 1 \text{ for some } \alpha < \omega_1 \text{ and } s(\alpha) = n\}.$$

Then each A_n is an antichain. Note that the injectivity of each $s \in A_n$ is used to show that A_n is an antichain.

Since $\bigcup_{n\in\omega} A_n$ covers the elements of T of successor length, at least one of A_n, $n \in \omega$ must be uncountable.

(2): Suppose that T is the tree as constructed in the proof of Theorem 8. Then $T_0 := T \cap T_{inj} \supseteq \{f_\alpha : \alpha < \omega_1\}$ is an ω_1-tree. Thus T_0 is not Suslin by (1) and hence T is neither Suslin. $\square$ (Lemma 9)

Proposition 10 (1) *If an iterable $\mathcal{P}$ contains all c.c.c. posets, then the $\mathcal{P}$-LgLCA for extendible implies $\neg\mathsf{CH}$ + SH. If $\mathcal{P}$ satisfies (f) then we have $2^{\aleph_0} = \aleph_2$ + SH. If $\mathcal{P}$ is cardinality and cofinality preserving then $2^{\aleph_0}$ is fairly large (at least weakly hyper-Mahlo) + SH.*

(2) *Suppose that $\mathcal{P}$ is an iterable class of posets whose elements are all productively c.c.c., and $\mathcal{P}$ contains $\mathrm{Fn}(\omega, 2)$. Then the $\mathcal{P}$-LgLCA for extendible implies $\neg\mathsf{CH}$ + $\neg\mathsf{SH}$. In this case, the continuum is fairly large (at least weakly hyper-Mahlo).*

(3) *The LgLCAA for hyperhuge implies $\diamondsuit_{\omega_1}$. In particular, this axiom implies CH + $\neg\mathsf{SH}$.*

(4) *For $\mathcal{P}$ = σ-closed posets, the $\mathcal{P}$-LgLCA for extendible implies CH + $\neg\mathsf{SH}$. Further, this axiom also implies $\diamondsuit_{\omega_1}$.*

(5) *For $\mathcal{P}$ = subcomplete posets, the same statements as those of (4) hold.*

Proof. (1): Since $\mathcal{P}$ contains all c.c.c. posets, $\mathcal{P}$-LgLCA implies $\neg\mathsf{CH}$ by Lemma 1. Since $\mathcal{P}$ contains all c.c.c. posets, $\mathsf{MA}(\omega_1)$ holds by Theorem 6. Thus SH follows.

(2): By Theorem 8, we have $\Vdash_{\mathrm{Fn}(\omega,2)}$ " there is a Suslin tree ". Since the statement "there is a Suslin tree" is Σ_2, and by Theorem 3, there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models$ " there is a Suslin tree". Since elements of $\mathcal{P}$ are productively c.c.c., it follows by Lemma 7, (1) that the Suslin tree in W remains Suslin in V.

(3): Suppose that LgLCAA for hyperhuge holds. Then we have $\diamondsuit_{\omega_1}$ (see Theorem 5.7, (2) in Fuchino [3]). Thus $\neg\mathsf{SH}$ holds (see e.g. Theorem 7.8 in Kunen [8]).

(4): Assume that the $\mathcal{P}$-LgLCA for extendible holds for $\mathcal{P}$ = σ-closed posets. Since "there is a Suslin tree" is Σ_2 and this statement can be forced by a σ-closed poset, Theorem 3 implies that there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models$ " there is a Suslin tree". By the choice of $\mathcal{P}$ and by Lemma 7, (2), the Suslin tree in W remains Suslin in V.

Since $\diamondsuit_{\omega_1}$ can be forced by σ-closed forcing, "$\diamondsuit_{\omega_1}$" is Σ_2, and a diamond sequence is preserved by σ-closed forcing, the same argument as above shows that $\mathcal{P}$-LgLCA for extendible implies $\diamondsuit_{\omega_1}$.

(5): The same argument as that of (4) shows the assertion. $\square$ (Proposition 10)

▶ **Open Problem (H.Sakai):** Is there a subclass $\mathcal{P}$ of semicompete posets such that $\mathcal{P}$-LgLCA implies $\mathsf{CH} + \mathsf{SH}$?

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