κ is generically measurable for a p.o. $\mathbb{P}$

$\Leftrightarrow$ there is $(V,\mathbb{P})$-generic G $\quad M, j \subseteq_{\text{class}} V[G]$ $\quad M \subseteq_{\text{transitive}} V[G]$

$j : V \xrightarrow{\prec} M \quad \mathrm{crit}(j) = \kappa$

κ is generically supercompact for a class $\mathcal{P}$ of p.o.s.

$\Leftrightarrow \forall \lambda \geq \kappa \ \exists \mathbb{P} \in \mathcal{P}$ for some $(V,\mathbb{P})$ generic G

$M, j \subseteq_{\text{class}} V[G] \quad M \subseteq_{\text{transitive}} V[G] \quad j : V \xrightarrow{\prec} M$

$\mathrm{crit}(j) = \kappa \quad j(\kappa) > \lambda$ and $j''\lambda \in M$

κ is Laver-generically supercompact (superhuge, super almost huge) for a class $\mathcal{P}$ of p.o.s.

$\Leftrightarrow \forall \lambda \geq \kappa \ \forall \mathbb{P} \in \mathcal{P} \ \exists \mathbb{Q} \in \mathcal{P}$ s.t. $\mathbb{P} \lessdot \mathbb{Q}$ there are

$(V,\mathbb{Q})$-gen H $\quad j, M \subseteq V[H] \quad M \subseteq_{\text{transitive}} V[H]$

$j : V \xrightarrow{\prec} M \quad \mathrm{crit}(j) = \kappa \quad j(\kappa) > \lambda \quad \mathbb{P}, H \in M$

$j''\lambda \in M.$

$j''\mu \in M$ for all $\mu < j(\kappa)$

$j''j(\kappa) \in M$

Lemma 1 If κ is gen. measurable for some $\mathbb{P}$ then κ is regular

Proof: Suppose not and let $f : \mu \to \kappa$ with $\mu < \kappa$ cofinal. Let $\mathbb{P}, G, j, M$ be as in the def of gen. meas.

Then $j(f) = f \quad V \models \kappa = \sup f$

By elementarity $M \models j(\kappa) = \sup j(f)$ [$= f$] $\Rightarrow j(\kappa) = \kappa$ $\lightning$ $\boxtimes$

Lemma 2 Suppose $j : V \xrightarrow{\prec} M \subseteq V[G]$, $\mathrm{crit}(j) = \kappa$, $j''\lambda \in M$. Then

(1) $\forall A \in V \ (|A| \leq \lambda \Rightarrow j''A \in M)$

(2) $j \restriction \lambda,\ j \restriction \lambda^2 \in M$ $\quad \lambda \times \lambda$

(3) For $A \in V$ $\ A \subseteq \lambda$ $\ A \subseteq \lambda^2$ we have $A \in M$

(4) $(\lambda^+)^M \geq (\lambda^+)^V$. Thus, if $(\lambda^+)^V = (\lambda^+)^{V[G]}$, then $(\lambda^+)^M = (\lambda^+)^V$

(5) $\mathcal{H}((\lambda^+)^V) \subseteq M$

(6) $j \restriction A \in M$ for $A \in \mathcal{H}((\lambda^+)^V)$

$f : \kappa \to V_\kappa$

$f(\kappa)$

Proof:

(1): $f : \lambda \xrightarrow{\text{surj}} A$

$j(f) \restriction j''\lambda \in M \quad \mathrm{im}(j(f) \restriction j''\lambda) \in M$ [$= j''A$]

(2): $(j \restriction \lambda)^{-1}$ = collapsing function of $j''\lambda$ [$\in M$]

$g : \lambda \xrightarrow{\text{bij}} \lambda^2 \in M$

(3): $A = j^{-1}{}''(j''A) \in M$ [$(j \restriction \lambda^2)^{-1}$]

(4): Suppose $\mu < (\lambda^+)^V$ then there is

Let $r \subseteq \lambda \times \lambda$ be s.t. $\mathrm{ot}(r) = \mu$ ...

(5): Suppose $a \in \mathcal{H}((\lambda^+)^V)$ $\ \mu \leq \lambda$

$\tilde{a} = \mathrm{trcl}(a) \cup \{a\}$ $\quad \langle \tilde{a}, \in \rangle \cong \langle \mu, E \rangle \in M$

(6): It is enough to show $j \restriction \mathrm{trcl}(A) \in M$

$\mathrm{trcl}(A) \in \mathcal{H}((\lambda^+)^V)$ $\quad A, \mathrm{trcl}(A) \in M \cap V$ by (5)

$j''\mathrm{trcl}(A),\ j''(\in \restriction \mathrm{trcl}(A)) \in M$

by (1)

$(j \restriction \mathrm{trcl}(A))^{-1}$ is Mostowski collapse of

$j''\mathrm{trcl}(A)$ Thus $j \restriction \mathrm{trcl}(A) \in M.$ $\boxtimes$

Thm 3 (?, ?) (1) consis(ZFC + $\exists$ supercompact (superhuge, super almost huge))

$\Rightarrow$ consis(ZFC + $\exists$ Laver gen. supercompact for σ-closed p.o.s. ($\mathcal{P}$, all))

(2) consis(ZFC + $\exists$ superhuge (super almost huge))

$\Rightarrow$ consis(ZFC + $\exists$ Laver gen. super almost-huge for proper posets (all, $\mathcal{P}$))

(3) consis(ZFC + $\exists$ supercompact (or superhuge))

$\Rightarrow$ consis(ZFC + $\exists$ Laver gen. supercompact (or super almost huge) for ccc posets ($\mathcal{P}$, all))

Prop 4 (1) If κ is ___-gen measurable for ω_1 pres. $\mathbb{P}$ then $\kappa > \omega_1$

(2) If κ is Laver-gen. supercompact for ω_1-preserving $\mathcal{P}$ with $Col(\omega_1, \{\omega_2\}) \in \mathcal{P}$ then $\kappa = \omega_2$

(3) If κ is Laver gen supercompact for $\mathcal{P}$ with $\mathbb{P} \in \mathcal{P}$ adding a new real. Then $\kappa \leq 2^{\aleph_0}$

(4) If κ is gen. supercompact for $\mathcal{P}$ s.t. all $\mathbb{P} \in \mathcal{P}$ do not add any reals then $2^{\aleph_0} < \kappa$

Proof (1): Suppose not. Then $\kappa = \omega_1$

Let G be $(V,\mathbb{P})$-gen $j : V \xrightarrow{\prec} M \subseteq V[G]$ s.t.

$crit(j) = \omega_1$ $M \models j(\kappa) = \omega_1$ $M \models \kappa$ is countable.

$V[G] \models$ ... A contradiction to the assumption on $\mathbb{P}$

(2): By (1) $\kappa \geq \omega_2$ Suppose $\kappa > \omega_2$.

Let $\mathbb{Q}$ be s.t. $Col(\omega_1, \{\omega_2\}) \leq \mathbb{Q}$ and let H be $(V,\mathbb{Q})$-gen $j : V \xrightarrow{\prec} M \subseteq V[H]$ $crit(j) = \kappa$

we $M \models \underbrace{j(\omega_2)}_{= \omega_2}$ is "ω_2" But $H \in M$

and $Col(\omega_1, \{\omega_2\})$ part of H collapses ω_2.

A contradiction.

(3): Suppose $\mu < \kappa$ and $\langle a_\alpha : \alpha < \mu \rangle$ be a sequence of reals.

It is enough to show $\langle a_\alpha : \alpha < \mu \rangle$ does not enumerate reals.

Let $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$ Let H be $(V,\mathbb{Q})$-gen. with

$j : V \xrightarrow{\prec} M \subseteq V[H]$ $crit(j) = \kappa$ | $M \models \langle a_\alpha \ \alpha < \mu \rangle$ doesn't enumerate $2^{\aleph_0}$

$j(\langle\ \rangle) = \langle\ \rangle$ | By elementarity

$V \models$ ...

Suppose $\kappa \leq 2^{\aleph_0}$

(4): Let $\lambda > 2^{\aleph_0}, \kappa$

Let $\mathbb{P} \in \mathcal{P}$ be s.t. for (generic) $(V,\mathbb{P})$-gen G

with $j, M \subseteq V[G]$

$j : V \xrightarrow{\prec} M \subseteq V[G]$ $crit(j) = \kappa$

$j(\kappa) > \lambda$

$M \models |(2^{\aleph_0})^M| \geq j(\kappa)$

$(2^{\aleph_0})^M = (2^{\aleph_0})^V$ $> \lambda > 2^{\aleph_0}$

A contradiction.

$V \models 2^{\aleph_0} \geq \kappa$

by elementarity

$j''trcl(A)$, $j''(\in \restriction trcl(A)) \in M$

by (1)

$(j \restriction trcl(A))^{-1}$ is Mostowski collapse of $j''trcl(A)$ Thus $j \restriction trcl(A) \in M$. ■

Thm 3 (a.?) (1) Con(ZFC + ∃ supercompact) super almost huge superhuge

⇒ Con(ZFC + ∃ Laver gen supercompact for σ-closed posets) ccc

(2) Con(ZFC + ∃ superhuge super almost huge)

⇒ Con(ZFC + ∃ Laver gen. super almost-huge for proper posets) ccc

(3) Con(ZFC + ∃ supercompact (or superhuge))

⇒ Con(ZFC + ∃ Laver gen. supercompact (or super almost huge) for ccc posets)